Cosmological Parameters from CMB

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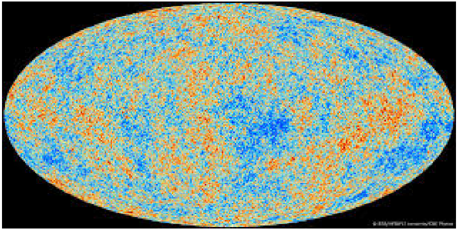
Outline

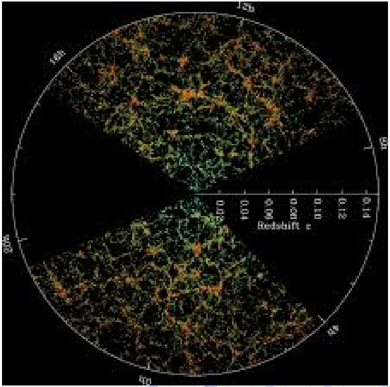
- Sound horizon r_d
- $\Omega_{\mathrm{M}} h^2$ and $\Omega_{\mathrm{B}} h^2$
- H_0
- Ω_k
- $\Omega_{\mathrm{M}} h^2$ and $\Omega_{\mathrm{B}} h^2$

Baryon Acoustic Oscillations=BAO

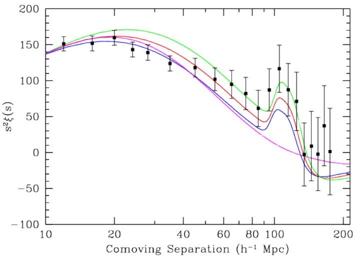
Acoustic oscillations in pre-recombination universe imprinted on

CMB anisotropies & Large-Scale Structure



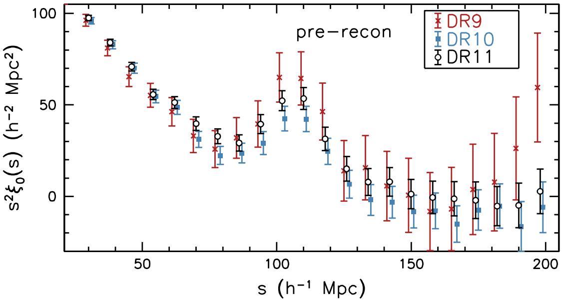


BAO Peak in galaxy correlation function



SDSS (2006)

SDSS-BOSS (2014)



Galaxies like to be separated by 105Mpc/h (147.5Mpc).

Acoustic waves $\Leftrightarrow$ $\sim$perfect fluids

"perfect fluid": mean-free-path of particles much less than spatial extent of perturbation.

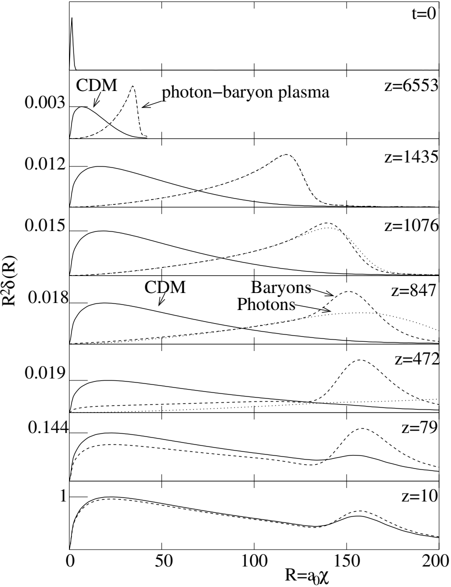
Early universe:
WIMPS: $\lambda_{mfp} \gg c/H(z)$ $\Rightarrow$no waves
photon-electron-proton plasma: $\lambda_{mfp} \ll c/H(z)$
electron-photon (Compton) scattering + electron-proton (Coulomb) scattering.

$\Rightarrow$ plasma supports acoustic waves until recombination.

$$c_s^s = \left(\frac{dP}{d\rho}\right)_{\mathrm{adiabatic}} \sim \frac{c^2}{3}$$

(since ρ is dominated by photons with $P = \rho/3$.)

A Universe with one perturbation



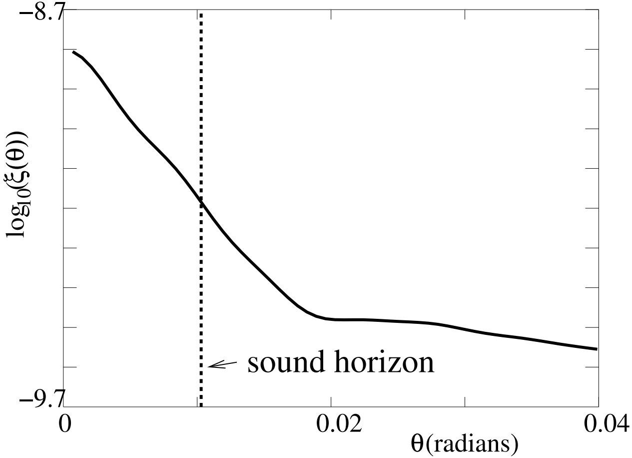
$t = 0$

$c_s \sim c/\sqrt(3)$
(γ,p,e plasma)

Wave stops at recombination
($r \sim 150\mathrm{kpc}$)

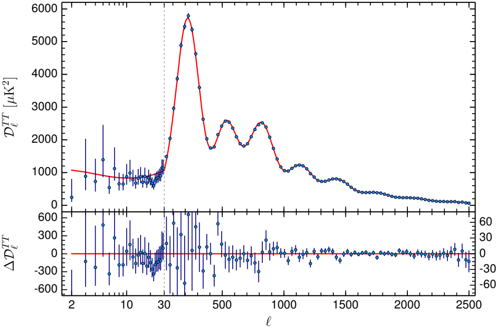
Today: Enhanced correlation
at $r = 147.5$Mpc

Temperature correlation function



Large correlations for $\theta < 2r_d/D(z_{rec})$
Small (primordial) correlations for $\theta > 2r_d/D(z_{rec})$

Planck CMB anistropy spectrum



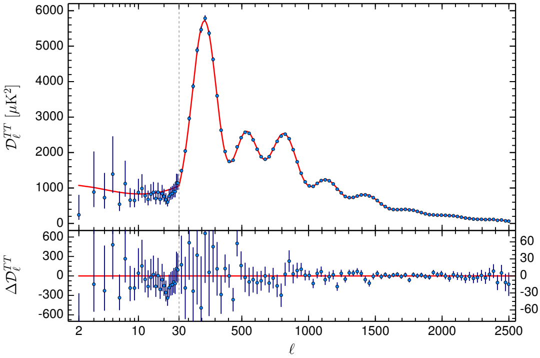
$$T(\theta,\phi) = \sum_{\ell,m} a_{\ell m} Y^{\ell}_{m}(\theta,\phi) \qquad D_\ell^{TT} = \frac{\ell(\ell+1)}{2\pi}\langle |a_{\ell m}|^2\rangle_m$$

The angular power spectrum, D_ℓ^{TT}:

$\sim$Fourier transform of correlation function $\xi(\theta)$

Measure of the anisotropy at angular scale $\sim \pi/\ell$.

CMB anistropy spectrum ⇒flat-ΛCDM parameters



Spectrum shape (relative peak heights) gives $\rho_{\mathrm{M}}/\rho_\gamma,\ \rho_{\mathrm{B}}/\rho_\gamma$
$\Rightarrow \Omega_{\mathrm{M}} H_0^2, \Omega_{\mathrm{B}} H_0^2$

$\Rightarrow r_d$

First peak ($\ell_1 \sim 200 \sim D(z=1060)/r_d$:

$$D(z) = \int_0^z \frac{dz}{\left[H_0^2 + \Omega_{\mathrm{M}} H_0^2[(1+z)^3 - 1]\right]^{1/2}} \qquad \Rightarrow H_0$$

($\sim 10\%$ of integral in H_0 dominated region)

Guesstimate of the sound horizon, r_d

$c_s \sim c/\sqrt{3}$ (relativistic plasma: $c_s^2 = p/\rho \sim 1/3$)

Age of universe at recombination $\sim$ 380000yr

$\Rightarrow r_d \sim 3.8 \times 10^5 \text{ly}/\sqrt{3}$ (at recombination, $z \sim 1060$)

$\sim$ 100kpc (at recombination, $z \sim 1060$))

$\sim$ 100Mpc (today)

Calculation of the sound horizon

Same as "particle horizon" except $c_s < c$
$c_s = (c/\sqrt{3}) f(\rho_{\rm B}/\rho_\gamma)$ (baryon inertia slows sound)

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)dz}{H(z)} = \sqrt{\frac{3}{8\pi G}} \int_{z_d}^{\infty} \frac{c_s(z)dz}{\sqrt{\rho_{\rm M} + \rho_\gamma + \rho_\nu}}$$

We normalized to the present photon density

$$r_d = \sqrt{\frac{3}{8\pi G \rho_\gamma(0)}} \int_{z_d}^{\infty} \frac{c_s(\rho_{\rm B}/\rho_\gamma)dz}{(1+z)^2\sqrt{\rho_{\rm M}/\rho_\gamma + 1 + \rho_\nu/\rho_\gamma}}$$

COBE gives us $\rho_\gamma(0)$ and the CMB spectrum shape (Planck) gives us the density ratios $\rho_{\rm M}/\rho_\gamma$, $\rho_{\rm B}/\rho_\gamma$, ρ_ν/ρ_γ.

r_d from COBE-Planck

Imposing three neutrino families ($\rho_\nu = 0.23 N_\nu \rho_\gamma$) gives

$$r_d = (147.5 \pm 0.6)\mathrm{Mpc}$$

Fitting the CMB spectrum for N_ν gives $N_\nu = 3.36 \pm 0.7$ and

$$r_d = (143.4 \pm 3.1)\mathrm{Mpc}$$

ΛCDM Cosmological Parameters

H_0: present expansion rate
Major components: $\Omega_\Lambda, \Omega_{\rm M}$, $\quad \Omega_k = 1 - \Omega_{\rm M} - \Omega_\Lambda \sim 0$
Minor Components: $\Omega_{\rm B}$ $(= \Omega_{\rm M} - \Omega_{\rm CDM})$,$\Omega_\nu, \Omega_\gamma$
(Ω's are present densities in units of $\rho_{\rm c} = 3H_0^2/8\pi G$)

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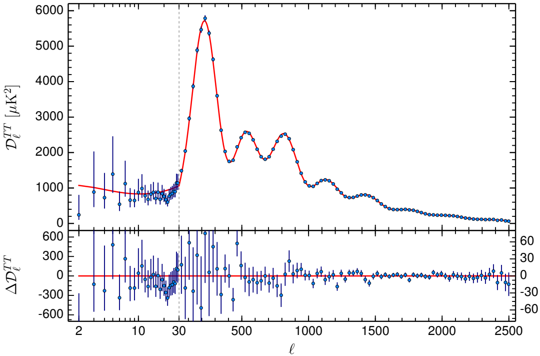
For flat ΛCDM $(\Omega_\Lambda = 1 - \Omega_{\rm M})$, only H_0 is still undetermined.

CMB anistropy spectrum



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$$D(z) = \int_0^z \frac{dz}{\sqrt{H_0^2 + \Omega_{\rm M} H_0^2[(1+z)^3 - 1]}} \qquad \Rightarrow H_0$$

($\sim 10\%$ of integral in H_0 dominated region)

Flat ΛCDM: CMB is enough

Planck 2015 (arXiv:1502.01589) (TT + LowP)

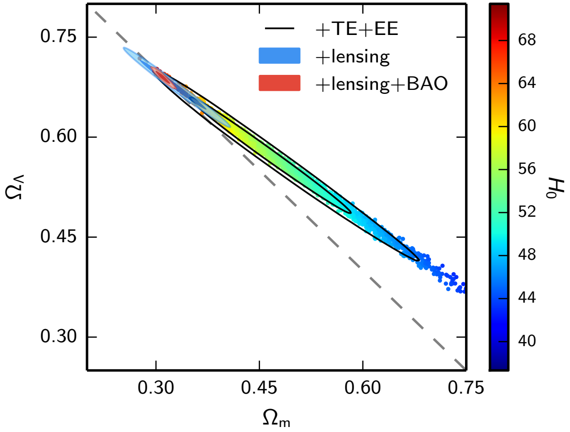
- $H_0 = 67.31 \pm 0.96$
- $\Omega_{\mathrm{M}} = 0.315 \pm 0.013$
- $\Omega_\Lambda = 1 - \Omega_{\mathrm{M}} = 0.685 \pm 0.013$
- $\Omega_{\mathrm{B}} h^2 = 0.02222 \pm 0.00023$

plus

- $A_s = (21.95 \pm 0.79) \times 10^{-10}$
  Amplitude of primordial scalar perturbations
- $n_s = 0.9655 \pm 0.0062$
  spectral index for scalar perturbations
- $\tau = 0.078 \pm 0.019$
  optical depth to last-scattering surface (reionization)

Non-flat ΛCDM: CMB not enough

$$D(H_0^2, \Omega_\mathrm{M} H_0^2) \;\rightarrow D_A(H_0^2, \Omega_\mathrm{M} H_0^2, \Omega_k H_0^2) \text{ or } D_A(H_0, \Omega_\mathrm{M}, \Omega_\Lambda)$$



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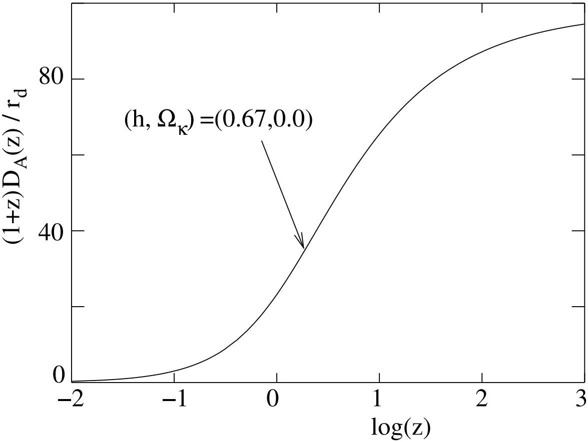
CMB Acceptable models need $(\Omega_\mathrm{M} h^2, \Omega_\mathrm{B} h^2)$
$= (0.142, 0.022)$
(spectrum shape)

$+$

Dots are combinations of $(\Omega_\mathrm{M}, \Omega_\Lambda)$ or (H_0, Ω_k) giving $r_d / D_A(1060) = 0.01041$

Three models that give the same $D_A(z = 1060)/r_d$



$(\Omega_{\rm M} h^2, \Omega_{\rm B} h^2)$
$= (0.142, 0.022)$
$r_d = 147.36$ Mpc

$(h, \Omega_k) = (0.67, 0.0)$

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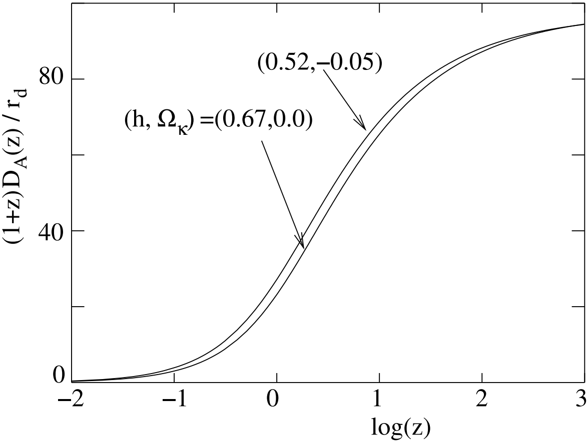

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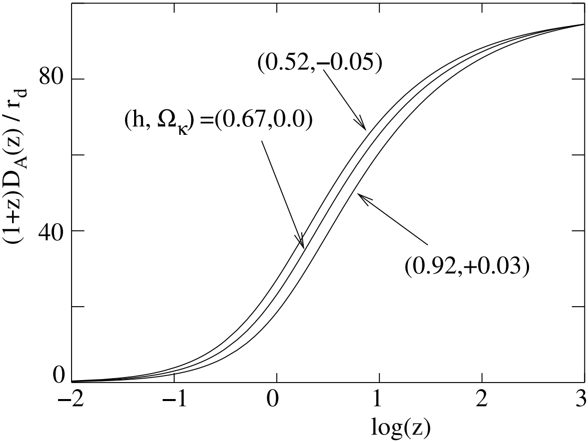

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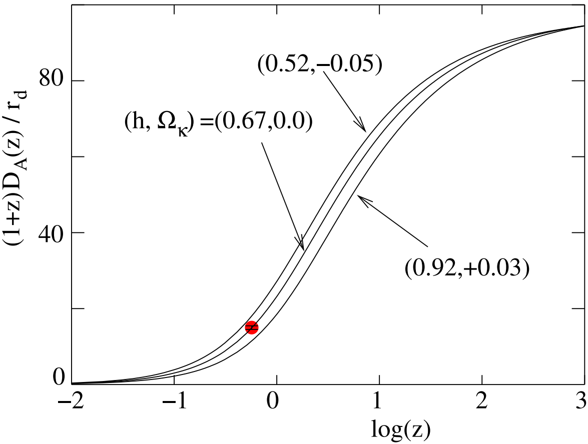
$(h, \Omega_k) = (0.92, 0.03)$

BAO at $z = 0.57$ picks flatness



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CMB + BAO: $\Omega_k = -0.0001 \pm 0.0054$

Extensions of flat ΛCDM that modify distance-redshift relation

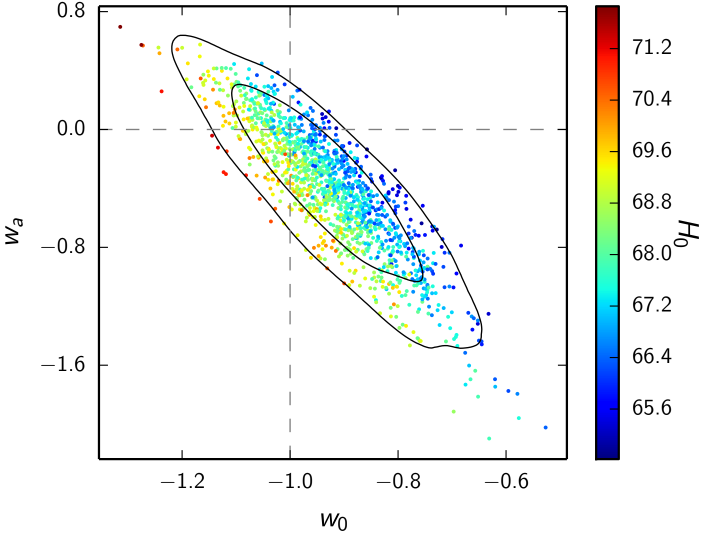
Examples of extensions:

- curvature ($\Omega_\Lambda + \Omega_{\mathrm{M}} \neq 0$)
- neutrino mass (relativistic→non-relativistic after recombination.)
- $w \neq -1$ ($w =$ dark energy presure/density)
- (w_0, w_a) (evolving $w(a) = w_0 + (1-a)w_a$)

Constraints requires CMB + something else

e.g. BAO, local H_0 meansurement, SNIa hubble diagram

Planck + BAO + SNIa: constraints on (w_0, w_a)



Structure formation

To understand the shape of the CMB spectrum we need to know a little bit about structure formation.

Here's how it's done:

- Expand $\rho(\vec{r}, t)$ in modes with comoving wavelengths
- Choose randomly conditions at Hubble entry for each mode amplitude
- Develop amplitude for each mode in time following ordinary differential equation (until non-linearities set in).

This is basically enough for CMB spectrum. To make structures like those observed (galaxies, clusters...), N-body techniques are generally used.

The modes for density and peculiar velocity fields

For each component i (CDM, baryons, photons, neutrinos....)

$$\rho_i(\vec{r,t}) = \bar{\rho}_i(t)\left[1 + \sum_{\vec{k}} \delta_{i,\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}\right] \qquad \vec{v}_i(\vec{r,t}) = \left[\sum_{\vec{k}} \vec{v}_{i,\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}\right]$$

$\delta_{i,\vec{k}}(t)$ coupled to $\vec{v}_i(\vec{r,t})$ via continuity equation: $\dot{\delta} \propto \vec{\nabla} \cdot \vec{v}$

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A useful quantity is the "gravitational potential" related to the density perturbations $\delta_{i,\vec{k}}(t)$ by the relativistic Poisson equation:

$$\phi(\vec{r,t}) = \left[\sum_{\vec{k}} \phi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}\right] \qquad \nabla^2\phi_{\vec{k}} \propto G\delta_{\vec{k}}$$

Co-moving modes are most useful

Unlike the usual relation, $\lambda_k = 2\pi/k$, in cosmology is is most useful to have modes with wavelengths that expand with the universe:

$$\lambda_k(t) = \frac{2\pi}{k}\frac{a(t)}{a_0}$$

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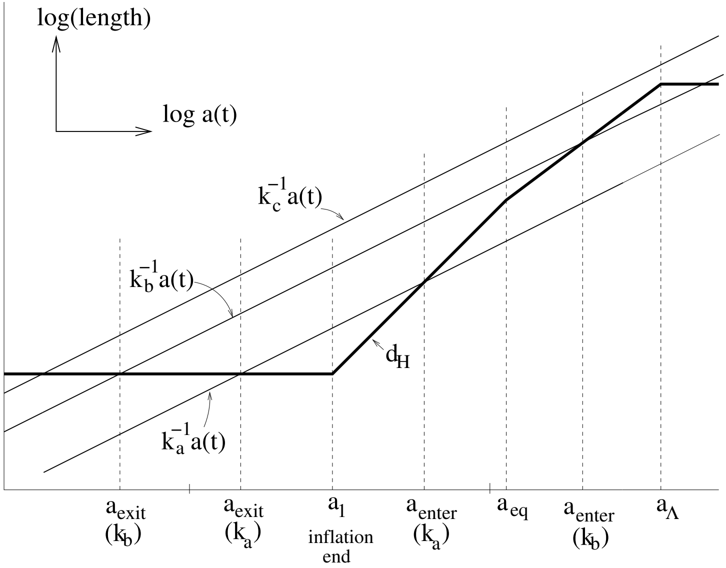
Since the Hubble radius, c/H, increases like a^2 in the radiation epoch and like $a^{3/2}$ during the matter epoch, each mode starts with a wavelength "outside" the Hubble radius and then "enters" the Hubble radius.

Modes leave and then enter the Hubble radius



Nearly scale-invariant Gaussian initial conditions

In the standard model, the potential fluctuations $\phi_{\vec{k}}(t_{enter})$ are Gaussian random varibles centered on zero. For scale-invariant spectra the width of the distribution is k-independent. The observed fluctuations have

$$\langle \phi_{\vec{k}}^2(t_{enter}) \rangle^{1/2} \sim 10^{-5}$$

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The factor 10^{-5} is present in a multitude of characteristics of our universe, most notably CMB temperature fluctuations, $\Delta T/T \sim 10^{-5}$ and the velocity dispersion of the largest galaxy clusters, $\langle v^2 \rangle / c^2 \sim 10^{-5}$. In inflationary models, these cosmology-size features were thus determined by the amplitude of the quantum fluctuations of the inflation field.

Adiabatic initial conditions

All species have the same initial fluctuations:

$$\delta_{CDM,\vec{k}}(t_{enter}) = \delta_{baryons,\vec{k}}(t_{enter}) = \delta_{\gamma,\vec{k}}(t_{enter}) = \delta_{\nu,\vec{k}}(t_{enter})$$

Predicted by the simplest inflationary models.

Time development of mode amplitudes

Matter epoch:

- density fluctuations grow: $\delta_{\vec{k}} \propto a(t)$
- potential fluctations constant: $\phi_{\vec{k}} \sim G\Delta M/\lambda \sim G\delta_{\vec{k}}\bar{\rho}\lambda^3/\lambda$
  This is why galaxy clusters "remember" the primordial potential fluctuations

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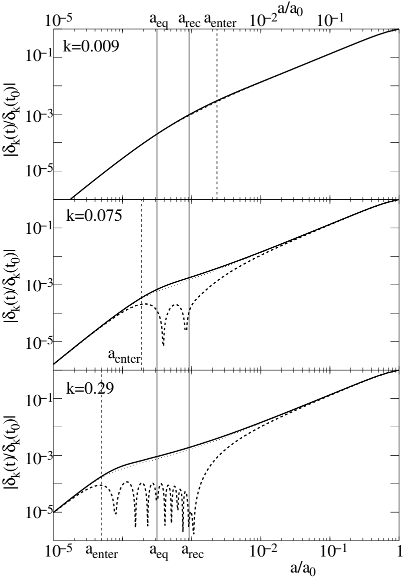
Radiation epoch:

- density fluctuations growth is inhibited
  baryon-photon plasma oscillates
  neutrinos free-stream
  growth of CDM fluctuations inhibited because gravity dominated by non-growing components (photons,neutrinos)
- potential fluctuations decay
  Ironically, this drives the baryon-photon oscillations, increasing temperature fluctuations.

$\delta(t)$ for long and short wavelenths



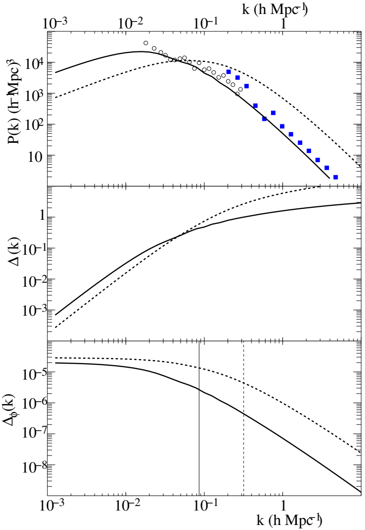
Three modes:

long-wavelength mode
No growth suppression

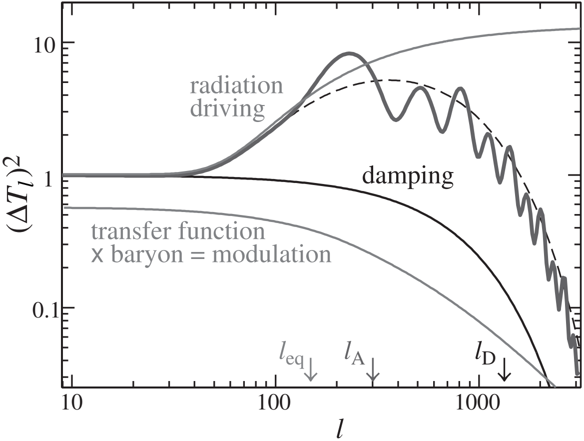
intermediate-wavelength mode

short-wavelength mode
radiation epoch:
 baryon oscillations and
 CDM growth suppression

Resulting power spectrum



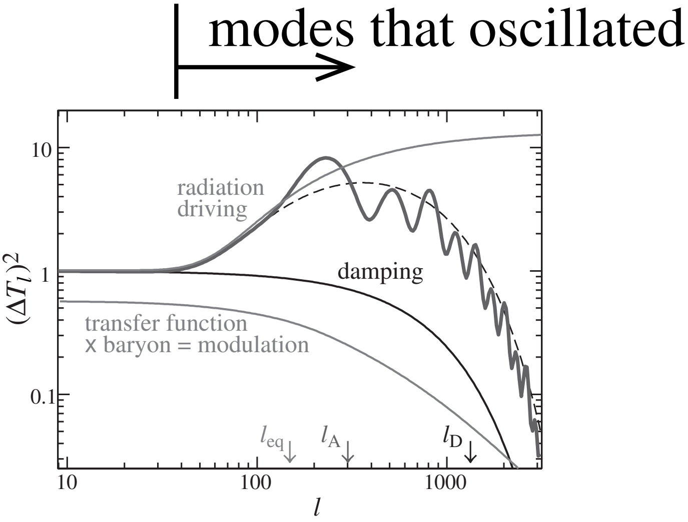
CMB spectrum has all modes



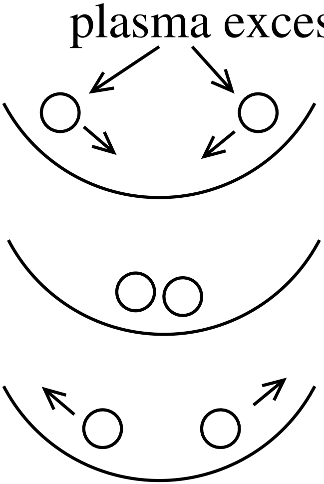
$$\ell \sim \frac{D_A(z_{rec})}{\lambda_{mode}(t_{rec})}$$

Three angular scales: $\ell_{eq}, \ell_A, \ell_D$ [Hu et al, (2001) ApJ 549,669]

Large $\ell \Rightarrow$ modes that oscillated



Baryon oscillations in CDM wells ($t < t_{rec}$)



plasma falls into well

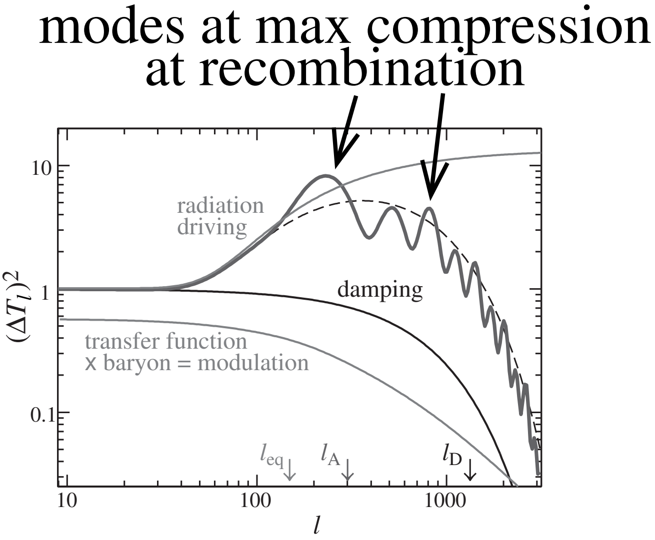
then

pressure stops fall

then

rebounds

Modes at extrema at recombination



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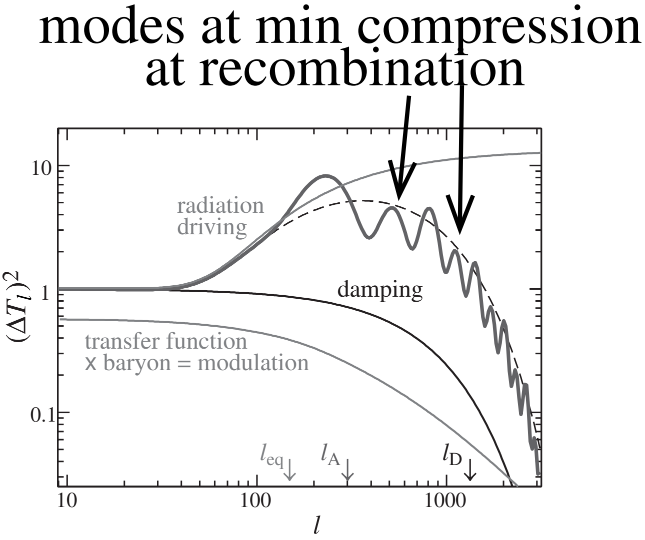
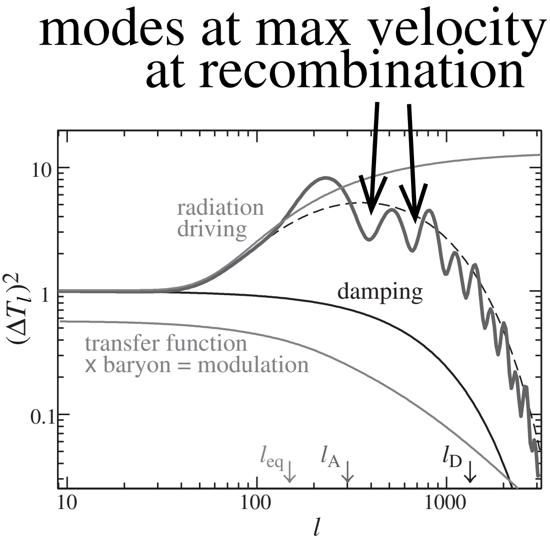
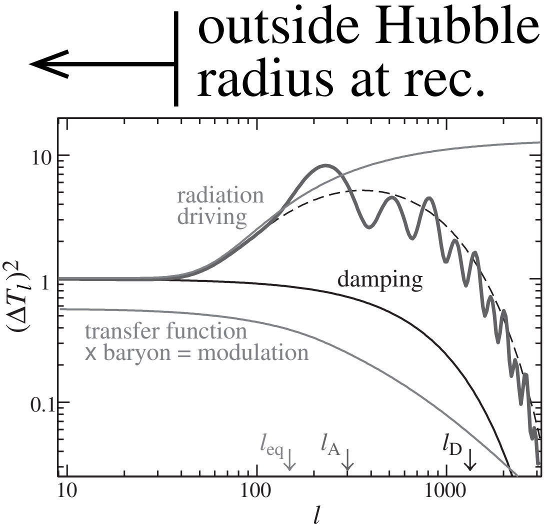


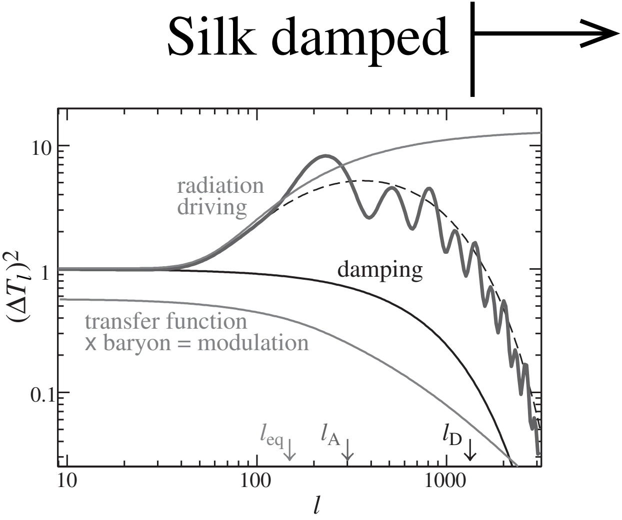
Doppler effect suppresed by baryon mass



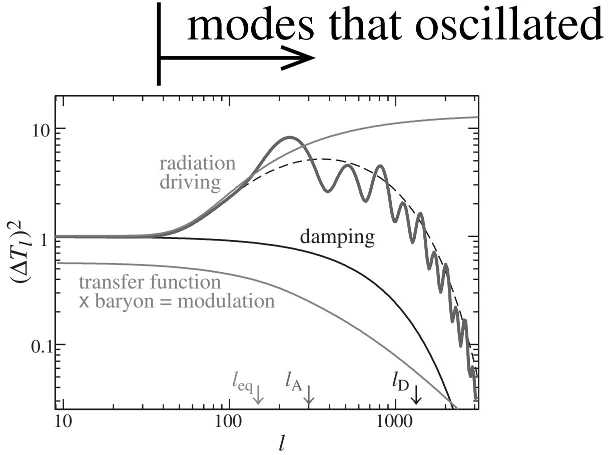
Low-ℓ modes $\Rightarrow$ primordial spectrum



High-ℓ modes damped by photon diffusion



Large $\ell \Rightarrow$ Potential decay $\Rightarrow$ radiation driving



$\Rightarrow$ amplitude of first peak increases with ℓ_A/ℓ_{eq}

ℓ_A/ℓ_{eq} depends on $\Omega_\gamma/\Omega_{\rm M}$

$$\ell_{eq} \sim \frac{D_A(z_{rec})}{(1+z_{eq})c/H(z_{eq})} \qquad \ell_A \sim \frac{D_A(z_{rec})}{r_d} \sim \frac{D_A(z_{rec})}{(1+z_{rec})c/H(z_{rec})}$$

$(1+z_{eq})c/H(z_{eq})$ is the wavelength that just fits inside the Hubble radius at the epoch of matter-radiation equality (z_{eq}). Wavelengths longer than this never oscillated.

$$H^2(z) \sim H_0^2[\Omega_{\rm M}(1+z)^3 + 1.66\Omega_\gamma(1+z)^4] \qquad \Rightarrow 1+z_{eq} = \frac{\Omega_{\rm M}}{1.66\Omega_\gamma}$$

ℓ_A/ℓ_{eq} depends on $\Omega_\gamma/\Omega_\mathrm{M}$

$$\ell_{eq} \sim \frac{D_A(z_{rec})}{(1+z_{eq})c/H(z_{eq})} \qquad \ell_A \sim \frac{D_A(z_{rec})}{r_d} \sim \frac{D_A(z_{rec})}{(1+z_{rec})c/H(z_{rec})}$$

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$$\ell_A/\ell_{eq} \propto \sqrt{1+z_{rec}}\left(\frac{1.66\Omega_\gamma}{2\Omega_\mathrm{M}}\right)^{1/2}$$

ℓ_A/ℓ_{eq} depends on $\Omega_\gamma/\Omega_{\rm M}$

$$\ell_{eq} \sim \frac{D_A(z_{rec})}{(1+z_{eq})c/H(z_{eq})} \qquad \ell_A \sim \frac{D_A(z_{rec})}{r_d} \sim \frac{D_A(z_{rec})}{(1+z_{rec})c/H(z_{rec})}$$

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$$\ell_A/\ell_{eq} \propto \sqrt{1+z_{rec}}\left(\frac{1.66\Omega_\gamma}{2\Omega_{\rm M}}\right)^{1/2}$$

Increasing $\Omega_\gamma/\Omega_{\rm M}$ increases radiation driving which increases peak heights

CMB spectrum shape determines $\Omega_{\mathrm{M}}h^2$ and $\Omega_{\mathrm{B}}h^2$

Peak heights determine $\ell_A/\ell_{eq} \sim \sqrt{\Omega_\gamma/\Omega_{\mathrm{M}}}$. Knowing $\Omega_\gamma h^2$ from COBE temperature measurement, we can then determine $\Omega_{\mathrm{M}}h^2$ to 1.4% precision:

$$\Omega_{\mathrm{M}}h^2 = 0.1426 \pm 0.0020 \quad \text{Planck arXiv1502.01589}$$

The photon-baryon ratio ($\Omega_\gamma h^2/\Omega_{\mathrm{B}}h^2$) determines the relative amplitudes of odd (compression) peaks and even (rarefaction) peaks, as well as the high-ℓ damping. This give 1% precision on $\Omega_{\mathrm{B}}h^2$:

$$\Omega_{\mathrm{B}}h^2 = 0.02222 \pm 0.00023 \quad \text{Planck arXiv1502.01589}$$

Note: primordial spectral index also affects relative peak heights:

$$n_s = 0.9655 \pm 0.0062 \qquad (\Delta T_\ell)^2 \propto \ell^{n-1}$$