Making Waves: Electromagnetic and Gravitational

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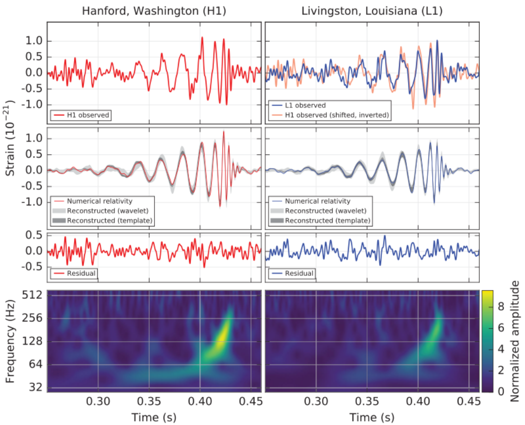
The Event



Outline:

- Gravity vs. Electromagnetism
- Quadrupoles vs. dipoles
- Ligo vs. quadrupole
- Beyond quadrupoles

Thanks to Nathalie Deruelle and Ed Porter!

Electromagnetism $\longrightarrow$ Gravitation

$A_\mu(x)$ coupled to $j_\mu(x)$
Maxwell:
$\Box^2 A_\mu \sim j_\mu$ (Lorenz gauge)
Charge conservation:
$\partial j^\mu / \partial x^\mu = 0$
Lorentz force:

$$\frac{d^2 x_\mu}{d\tau^2} \sim (q/m) F_{\mu\nu} \frac{dx_\mu}{d\tau}$$

$g_{\mu\nu}(x)$ coupled to $T^{\mu\nu}(x)$
Einstein:
$G_{\mu\nu} \sim G T_{\mu\nu}$
$(E, \vec{p})$ conservation:
$\partial T^{\mu\nu} / \partial x^\mu = 0$
Tidal forces:

$$\frac{D^2 \delta_\mu}{D\tau^2} \sim R^\mu_{\alpha\beta\gamma} \frac{D\delta_\alpha}{D\tau} \delta_\beta \frac{D\delta_\gamma}{D\tau}$$

Gravity has more indices than electromagnetism. Why?
(This is one reason it's more difficult)

Non-relativistic limits

Poisson eqn. for electrostatic potential

$$\nabla^2 A_0 = \rho_q$$

Potential far from a localized charge distribution:

$$A_0 \sim \frac{\int \rho_q dV}{r}$$

Poisson eqn. for gravitational potential

$$\nabla^2 \phi = 4\pi G \rho_m$$

Potential far from a localized mass distribution:

$$\phi \sim \frac{G \int \rho_M dV}{r}$$

Expect that relativistic generalization of Newton's theory will involve the energy density ρ_E ($= c^2 \rho_M$ in rest frame).

Lorentz Transformation of ρ_q and ρ_E

Consider a box (volume L^3) containing
N particles at rest, each of (mass,charge) $= (m, q)$
Boost by v, $\gamma = 1/\sqrt{1 - v^2/c^2}$:
$N \rightarrow N$
$q \rightarrow q$
$mc^2 \rightarrow \gamma mc^2$ (kinetic energy now $\neq 0$)
$L^3 \rightarrow \gamma^{-1} L^3$ (Lorentz contraction)

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$\rho_E = Nmc^2/L^3 \longrightarrow \gamma^2 \rho_E$ (00th component of a tensor: T_{00})

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Electromagnetism: J_μ is source of a vector field A_μ
Gravitation: $T_{\mu\nu}$ is source of a tensor field $g_{\mu\nu}$
(That's good: allows for equivalence principle through geodesic eqn.)
but
Gravitation is more complicated than electromagnetism because energy is more complicated than charge.

j^μ and $T^{\mu\nu}$ for a collection of particles near $(\vec{r}, t)$

$n(\vec{r}, t)$ = particles per unit volume;
(m, q)=mass, charge per particle.:

$$j^\mu(\vec{r}, t) \;=\; qn(\vec{r}, t)\left\langle \frac{p^\mu}{p^0} \right\rangle \qquad T^{\mu\nu}(\vec{r}, t) \;=\; n(\vec{r}, t)\left\langle \frac{p^\mu p^\nu}{p^0} \right\rangle$$

$$j^\mu \;=\; qn(\vec{r}, t)\begin{pmatrix} 1 \\ \langle\beta_x\rangle \\ \langle\beta_y\rangle \\ \langle\beta_z\rangle \end{pmatrix}$$

$$\beta \ll 1 \;\Rightarrow\; T^{\mu\nu} \rightarrow mc^2 n(\vec{r}, t)\begin{pmatrix} 1 & \langle\beta_x\rangle & \langle\beta_y\rangle & \langle\beta_z\rangle \\ \langle\beta_x\rangle & \langle\beta_x\beta_x\rangle & \langle\beta_x\beta_y\rangle & \langle\beta_x\beta_z\rangle \\ \langle\beta_y\rangle & \langle\beta_y\beta_x\rangle & \langle\beta_y\beta_y\rangle & \langle\beta_y\beta_z\rangle \\ \langle\beta_z\rangle & \langle\beta_z\beta_x\rangle & \langle\beta_z\beta_y\rangle & \langle\beta_z\beta_z\rangle \end{pmatrix}$$

Fields from small and distant sources

Static charge distribution:

$$A^0(r) =\sim \frac{\int \rho_q dV}{r}$$

Static mass distribution

$$\phi(r) =\sim \frac{G \int \rho_M dV}{r}$$

Fields from small and distant sources

Static charge distribution:

$$A^0(r) =\sim \frac{\int \rho_q dV}{r}$$

Time-dependent current distribution:

$$\vec{A}(r,t) \sim \frac{\int \vec{J}_\perp(\vec{r}', t-r)dV'}{r}$$

(dipole radiation)

Static mass distribution

$$\phi(r) =\sim \frac{G\int \rho_M dV}{r}$$

Time-dependent mass distribution:

$$h_{ij}(r,t) \sim G\frac{\int T_{ij,\perp}(\vec{r}', t-r)dV'}{r}$$

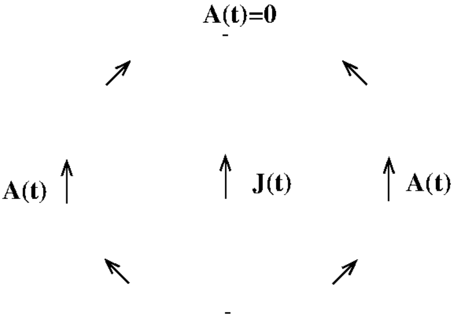
(quadrupole radiation)

Require: $r \gg \lambda \gg$ source size $\Rightarrow$ phase coherence over source

.

note: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and ϕ/c^2 are dimensionless.

$\vec{A}(r, \theta, t)$ sees only projected $\vec{J}$



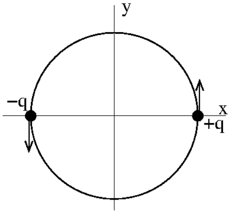
$\vec{E}$ and $\vec{B}$ transverse to $\vec{k}$
(consequence of $\vec{\nabla} \cdot \vec{E}, \vec{B} = 0$)

Only projected $\vec{J}$ contributes to wave.

Same is true for gravitational waves but more difficult to prove.

2 particles ($\pm q, m$) orbiting in xy plane

$x = \pm R \cos \omega t \quad v_x = \mp V \sin \omega t$
$y = \pm R \sin \omega t \quad v_y = \pm V \cos \omega t$
with $V = \beta c = R\omega \ll c$



$$\int j^\mu dV \;=\; 2q \begin{pmatrix} 0 \\ -\beta \sin \omega t \\ \beta \cos \omega t \\ 0 \end{pmatrix}$$

$$\int T^{\mu\nu} dV \;=\; 2m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta^2 \sin^2 \omega t & -\beta^2 \cos \omega t \sin \omega t & 0 \\ 0 & -\beta^2 \cos \omega t \sin \omega t & \beta^2 \cos^2 \omega t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Need to subtract off time averages to get source of waves.

2 particles ($\pm q, m$) orbiting in xy plane

$$x = \pm R \cos \omega t \qquad v_x = \mp V \sin \omega t$$
$$y = \pm R \sin \omega t \qquad v_y = \pm V \cos \omega t$$

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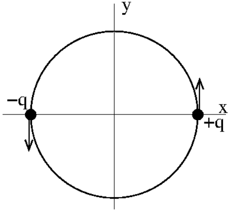


$$\int j^\mu dV \;=\; 2q \begin{pmatrix} 0 \\ -\beta \sin \omega t \\ \beta \cos \omega t \\ 0 \end{pmatrix}$$

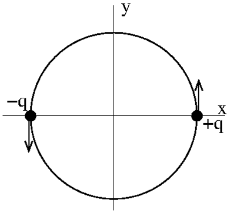
$$\int T^{\mu\nu} dV \;=\; m \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta^2 \cos 2\omega t & -\beta^2 \sin 2\omega t & 0 \\ 0 & -\beta^2 \sin 2\omega t & \beta^2 \cos 2\omega t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Gwaves have twice the frequency of the EMwaves (not surprising).

Fields far from source on z axis

$$x = \pm R\cos\omega t \qquad v_x = \mp V\sin\omega t$$
$$y = \pm R\sin\omega t \qquad v_y = \pm V\cos\omega t$$

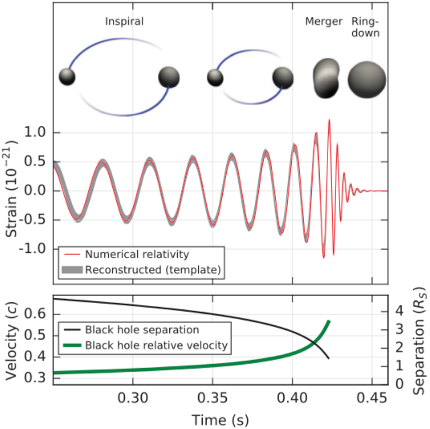
with $V = \beta c = R\omega \ll c$



$$A_\mu(z, t + r/c) \;=\; \frac{2q}{r}\begin{pmatrix} 0 \\ -\beta\sin\omega t \\ \beta\cos\omega t \\ 0 \end{pmatrix}$$

$$h_{\mu\nu}(z, t + r/c) \;=\; \frac{Gm}{r}\frac{1}{c^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta^2\cos 2\omega t & -\beta^2\sin 2\omega t & 0 \\ 0 & -\beta^2\sin 2\omega t & \beta^2\cos 2\omega t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

LIGO event waveform: frequency



$m_1 = 36M_\odot$, $m_2 = 29M_\odot$
$R_S \sim 2G \times 70M_\odot/c^2 \sim 200\text{km}$

.

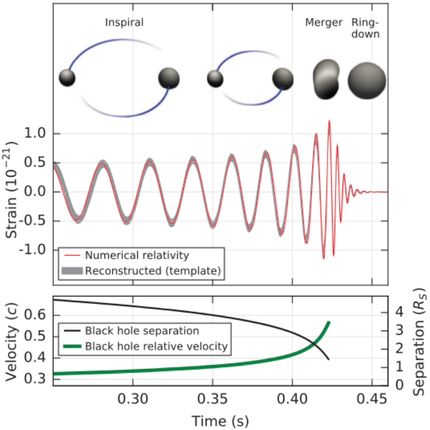
Kepler:

$$\tau^2 = \frac{4\pi^2(separation)^3}{G(m_1+m_2)}$$

$\Rightarrow \nu_{wave} = 2\nu_{orbit} = 29.2\text{Hz}$
at $separation = 5R_S$, $\beta \sim 0.3$
Observed=32.2Hz (not so bad)

Maximum frequency determines M:
$\tau(R_S) \sim GM/c$ (before redshift)

LIGO event waveform: strain



$m_1 = 36M_\odot$, $m_2 = 29M_\odot$
$r \sim 410\text{Mpc}$

.

$$Strain \sim h \sim \frac{GM}{rc^2}\beta^2 \sim 3\times10^{-21}\beta^2$$

Antenna response

Free electron:
Force$=e\vec{E}_x = e\partial_t A_x \sim \omega A_x$
Electron position oscillates with amplitude:

2 test particles separated by L
Tidal acceleration $\sim L\partial^2 h_{xx}/\partial t^2$
Separation oscillates with amplitude

$$\Delta x_e \sim (e/m_e)\omega^{-1}\frac{2q}{r}\beta \qquad\qquad \Delta x_{tp} \sim L\frac{Gm}{rc^2}\beta^2$$

$$\frac{\Delta x_e}{\Delta x_{tp}} \sim \frac{eq}{Gmm_e}\frac{\lambda_{gw}}{L}\beta^{-1}$$

Suppose gravitationally bound: $Gm_{BH}/\lambda_{gw} \sim \beta^3 \Rightarrow$

$$\frac{\Delta x_e}{\Delta x_{tp}} \sim \frac{\alpha_{em}\hbar c}{m_e c^2 L}\beta^{-4}\frac{q_{BH}}{e} \sim 3\times 10^{-18}\beta^{-4}\frac{q_{BH}}{e}$$

~ 1 for charge excess of $\sim 10^{18}/10^{58} \sim 10^{-30}$

Energy density of field

Static electric field, source q:

$$\rho_E \sim |\vec{\nabla} A_0|^2 \sim \frac{q^2}{r^4}$$

Static grav. field, source m:

$$\rho_E \sim G^{-1}|\vec{\nabla}\phi|^2 \sim \frac{Gm^2}{r^4}$$

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EMwave, source $\vec{J}(t)$:

$$\rho_E \sim |\partial_t \vec{A}|^2 \sim \left(\frac{\omega q \beta}{r}\right)^2$$

Gwave, source $T_{xy}(t)$

$$\rho_E \sim G^{-1}|\partial_t h|^2 \sim G\left(\frac{\omega m \beta^2}{r}\right)^2$$

Energy loss from radiation

Energy loss per orbital period $=$ Field energy in shell of thickness $\sim \lambda$.

EMwave, source $\vec{J}(t)$:

$$\rho_E \sim |\partial_t \vec{A}|^2 \sim \left(\frac{\omega q \beta}{r}\right)^2$$

$$\Delta E \sim (\omega q \beta)^2 \omega^{-1}$$

$$\frac{q^2}{R} \sim v^2 \;\Rightarrow\; \frac{\Delta E}{mv^2} \sim \beta^3$$

$\Rightarrow$ Hydrogen atom radiates it's energy in $\sim 10^6$orbits.

Gwave, source $T_{xy}(t)$

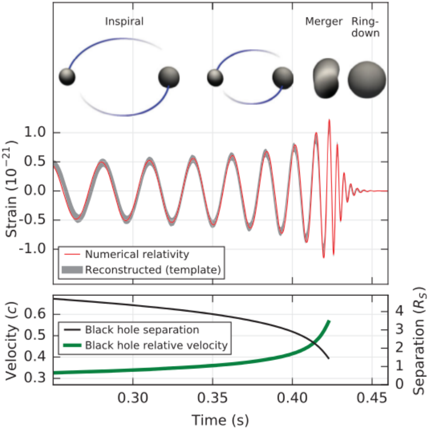
$$\rho_E \sim G^{-1}|\partial_t h|^2 \sim G\left(\frac{\omega m \beta^2}{r}\right)^2$$

$$\Delta E \sim G\left(\omega m \beta^2\right)^2 \omega^{-1}$$

$$\frac{Gm}{R} \sim v^2 \;\Rightarrow\; \frac{\Delta E}{mv^2} \sim \beta^5$$

$\Rightarrow$ Earth radiates it's energy in $\sim 10^{20}$orbits.

LIGO event waveform: energy loss by radiation

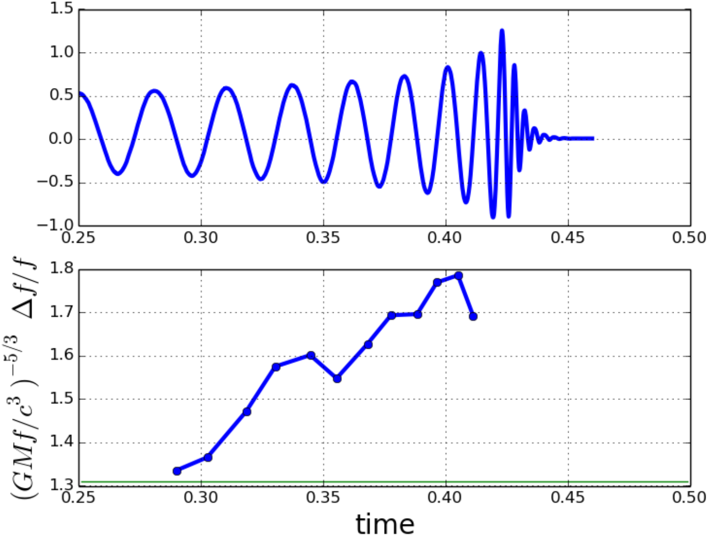


Energy loss $\Rightarrow$ frequency increases with time.
Frequency change, Δf in one orbital period:

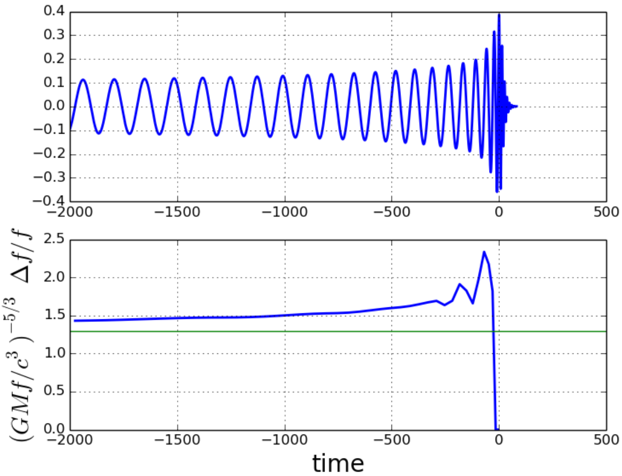
$$\frac{\Delta f}{f} = \frac{12}{5}(4\pi)^{8/3}(GMf)^{5/3}$$

(quadrupole formula, two equal masses $m_1 = m_2 = M$)
$\Rightarrow f^{-5/3}\Delta f/f$ time and f-independent according to quadrupole formula

Ligo event: $f^{-5/3}\Delta f/f$ increases with time

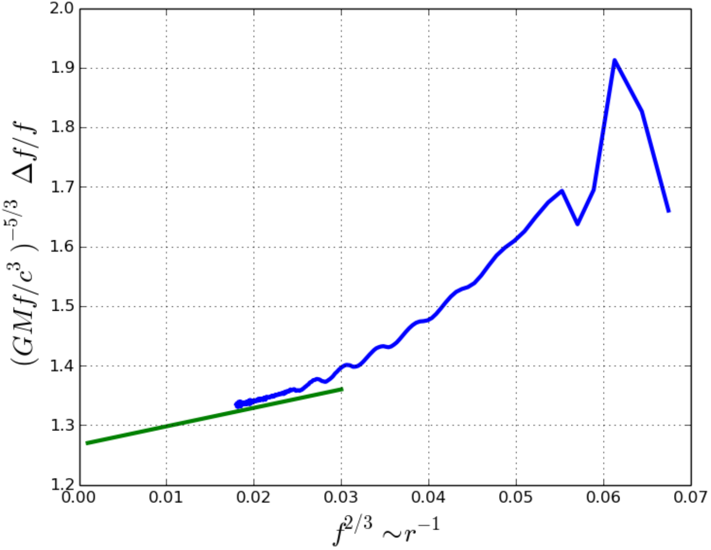


A simulated BH coalescence



Ligo sees only the last few orbits where quadrupole formula breaks down (relativistic corrections ⇒ octopole.....)

BH coalescence: extrapolation to $r = \infty$



$\Rightarrow$ Energy loss = quadrupole plus order GM/R

Beyond quadrupole radiation

Standard quadrupole calculation is fundamentally perturbative:

- calculate orbit (Newtonian or geodesic orbit for test particle)
- calculate radiation field in quadrupole approx. ($r \gg \lambda \gg R$)
- calculate backreaction on orbit and correct it.

Holistic approach: Integrate Einstein equation for $GM/R \ll 1$ (PPN):

- Geodesic eqn. not necessary
- Calculation is subtle because background space-time is not stable as in electrodynamic calculations (backreaction between orders of GM/R!)
- Include order unity dimensionless effects: orbital elipticity; mass ratio (non trivial in GR); spins.

Numerical Relativity for $GM/R \rightarrow 1$

Wikipedia on Numerical Relativity

In the puncture method the solution is factored into an analytical part, which contains the singularity of the black hole, and a numerically constructed part, which is then singularity free.....
Until 2005, all published usage of the puncture method required that the coordinate position of all punctures remain fixed during the course of the simulation. Of course black holes in proximity to each other will tend to move under the force of gravity, so the fact that the coordinate position of the puncture remained fixed meant that the coordinate systems themselves became "stretched" or "twisted," and this typically lead to numerical instabilities at some stage of the simulation.....
In 2005 researchers demonstrated for the first time the ability to allow punctures to move through the coordinate system, thus eliminating some of the earlier problems with the method. This allowed accurate long-term evolutions of black holes.

Summary

- Gravity is harder than Electrodynamics because it has more indices, but the standard first-order formulas are very similar.
- Gravity is harder than Electrodynamics because, rather than having particles obeying a force-equation in a fixed space-time, singularities of a time-varying metric move obeying the non-linear Einstein equations.
- Ligo sees only the "difficult" part of the event, so understanding the Ligo event means understanding why we can't.
- Lower-mass events will have the part where the quadrupole formula almost works (GM/r corrections). Unfortunately, the corrections are very difficult to calculate and unlikely to enter the textbooks.