Lecture #5

Electromagnetic Forces and Stresses

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“Ballpark” Estimate

- “Ballpark” Estimate
- Force in Single Turn or Turns Acting Independently
- Stress analysis in *ideal* solenoid—analytical approach
- Forces in ideal dipole
- Forces in ideal quadrupole
- Axial Forces

"Ballpark" Estimate

Analytical technique to enable *any* specialist of a design team to compute *ballpark* values of *any* key magnet parameters

This aim identical to those of my textbooks, 1st Edition(1994) & 2nd Edition (2009)

Leave it to a *specialist*, armed with sophisticated codes, for an *exact* (hopefully *correct*) *value*

An Example of an Expert's Error (Stress Computation)

Hoop Stress in an Infinite Solenoid

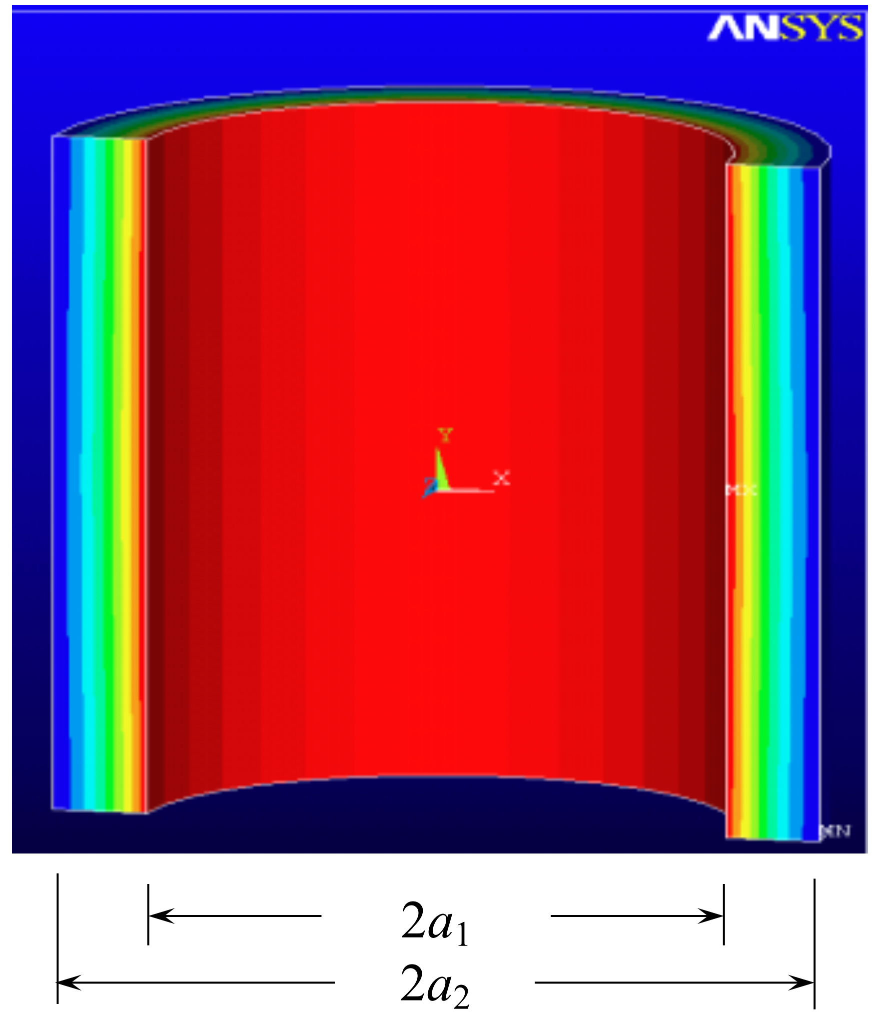
"Thin" and Infinite Coil:

$2a_1 = 0.5$ m;

$2a_2 = 0.7$ m;

$\alpha \equiv a_2 / a_1 = 1.4$;

$\lambda J = 5 \times 10^7$ A/m^2



Hoop Stress in an Infinite Solenoid (continuation)

"Thin" and Infinite Coil

$$B_\circ = \mu_\circ \lambda J a_1 (\alpha - 1)$$

$$= (4\pi \times 10^{-7}\,\mathbf{H/m})(5\times 10^7\,\mathbf{A})(0.25\,\mathbf{m})(0.4)$$

$$= 6.28\,\mathrm{T}$$

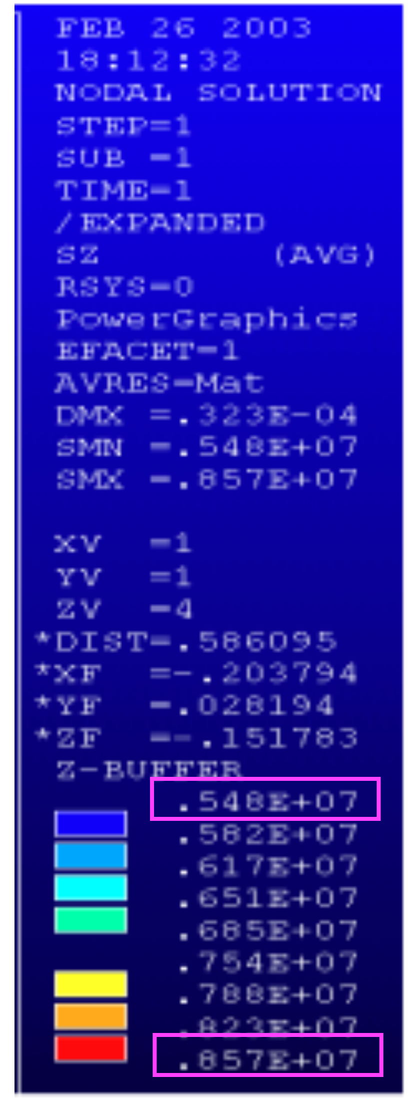
$$\tilde{\sigma}_\theta \simeq a_1 \left(\frac{\alpha+1}{2}\right) \lambda J \tilde{B}_z$$

$$= (0.25\,\mathbf{m})(1.2)(5\times 10^7\,\mathbf{A/m^2})(3.14\,\mathrm{T})$$

$= \quad 0.47 \times 10^8$ Pa

Average hoop stress

CODE



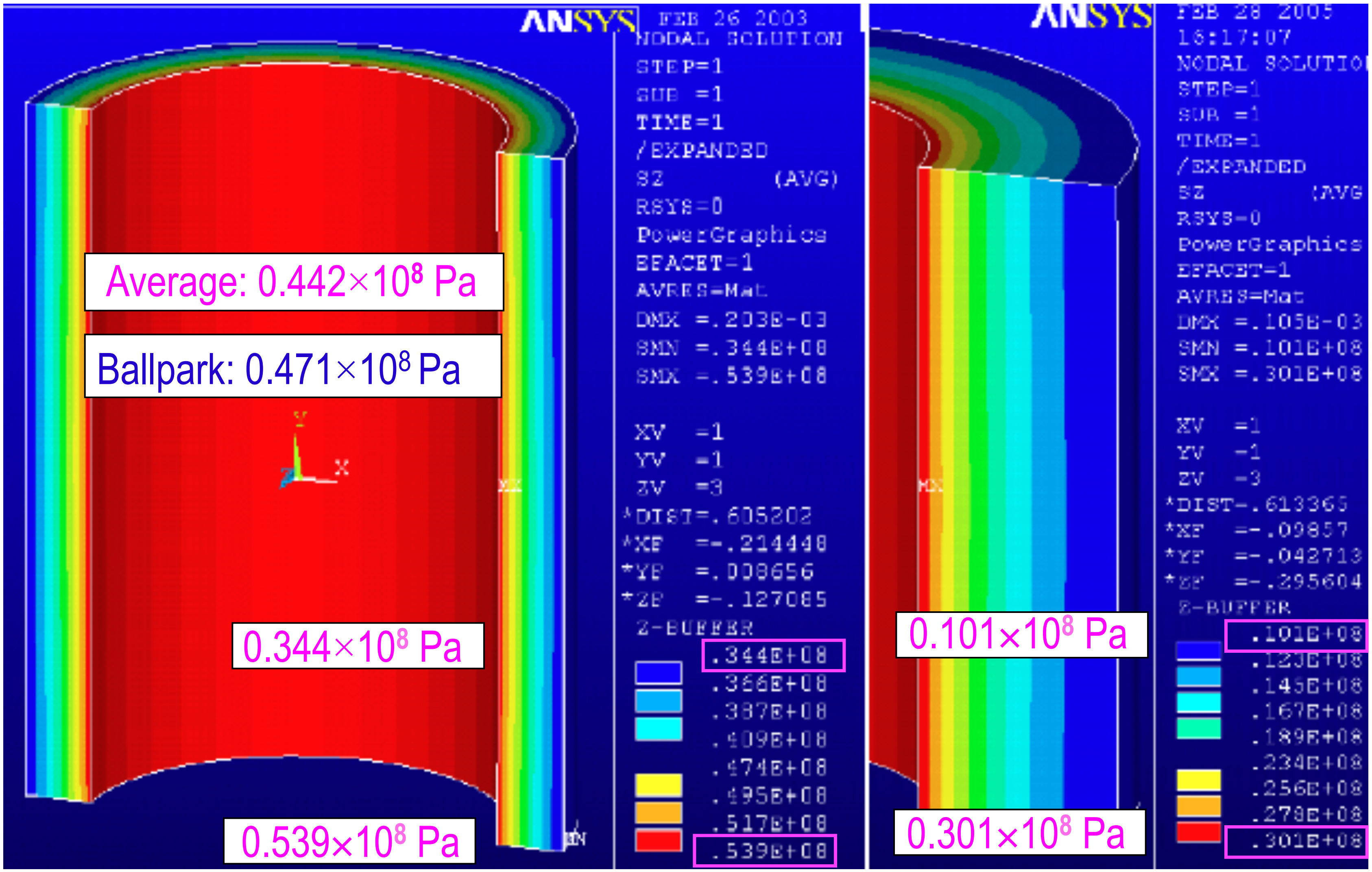
0.548×10^7 Pa

Average: 0.703×10^7 Pa ?????

0.857×10^7 Pa

Hoop Stress in an Infinite Solenoid

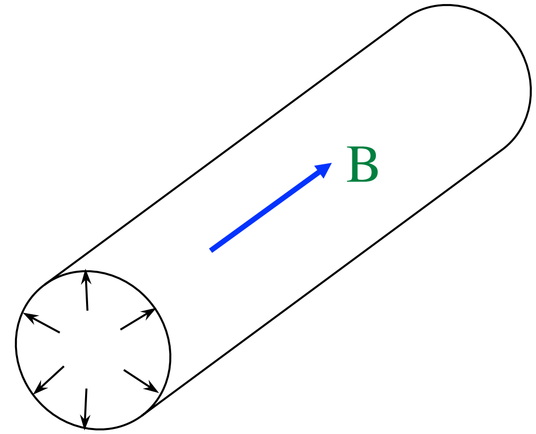
"Thin" "Thick"



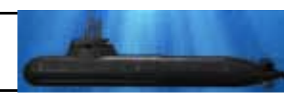
Force

For a solenoid, energy stored in the magnetic field acts equivalent to an internal pressure:

$$\frac{E_m}{\mathbf{Volume}} = \frac{B^2}{2\mu_\circ} = P_m$$



Undersea Depth [m]	P_m [atm]	B [T]	f [GHz]	*Remarks*
300	30	2.7	0.12	Maximum for submarines
11,000	1,100	16.5	0.7	Deepest sea bottom: *Challenger Deep*
22,100	2,210	23.5	1.0	High-strength stainless steel yields at 14,000 atm
400,000	40,000	100	4.26	



Force

Force can:

- Break the structure and destroy the magnet
- Damage insulation
- Damage superconductor, e.g., overstraining: brittle Nb_3Sn & HTS
- Degrade magnet performance:
  motion → release energy → quench → training (?)

Force Density

$$\vec{f} = \vec{J} \times \vec{B}$$

Compute $\vec{B}(x, y, z)$ throughout the winding volume

$$\left[\frac{\mathrm{N}}{\mathrm{m}^3}\right] = \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right] \times [\mathrm{T}]$$

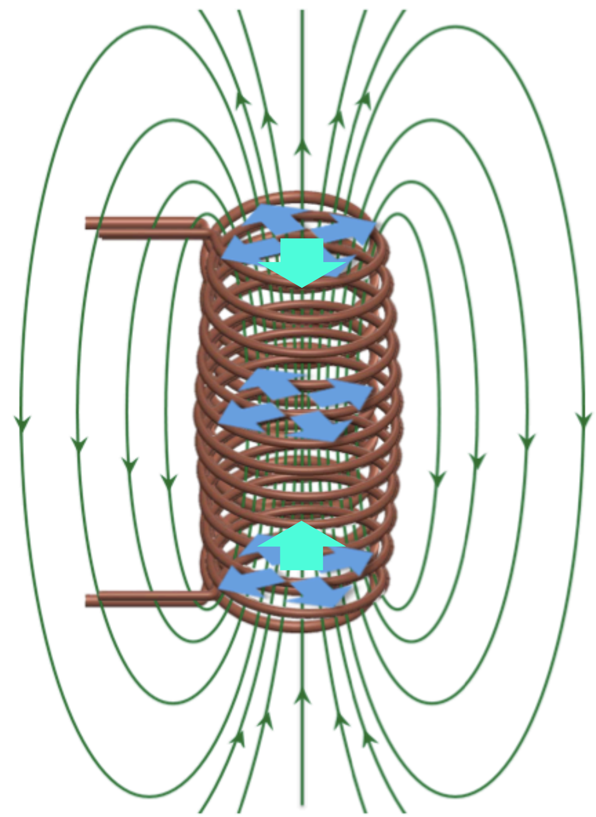
$$= \left[\frac{\mathrm{A}}{\mathrm{m}^2}\right] \times \left[\frac{\mathrm{V\,s}}{\mathrm{m}^2}\right] = \left[\frac{\mathrm{J}}{\mathrm{m}^4}\right] = \left[\frac{\mathrm{N\,m}}{\mathrm{m}^4}\right] = \left[\frac{\mathrm{N}}{\mathrm{m}^3}\right]$$

Combine computation & analysis with calculation of

- Usually done with a code
- Simple analytical figures (“ballpark”) in early design stages.
- Must have knowledge of materials properties:
  - Mechanical
  - Thermal (coefficient of expansion for thermal stresses) from RT to 4 K

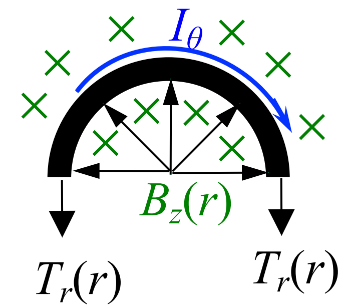
Forces and Stresses in Solenoids—General Consideration

- Radially expanding
  - σ_r *shows up as hoop stress,* σ_h *, in the conductor*
  - σ_r *must be kept negative in the winding to keep the turns from separating*
- Axially squeezing (compressive)



Force in Single Turn or Turns Acting Independently

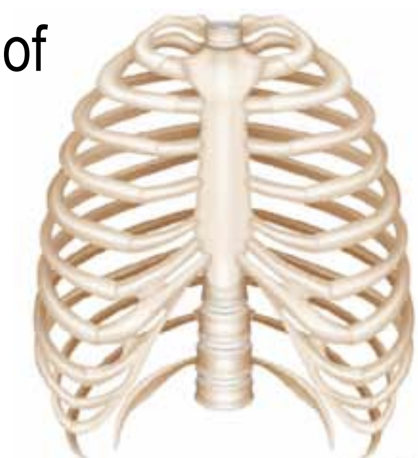
- Tension: $T_r(r) = rI_\theta \times B_z(r)$
- Overall hoop stress: $\sigma_\theta(r) = rJ_\theta \times B_z(r)$



Both J_θ and σ_θ averaged over the winding pack

Stresses increase with B, J, and r (size) $\Rightarrow$ $\boldsymbol{R\ I\ B}$ [N/m^3]

To remember a force (density) formula RIB, think of



Caution

Note that $rJ \times B$ applicable *only for single-turn coil*.

Many people still *mistakenly* use this for *multi-turn solenoids:* DO NOT!

Stress Analysis in *Ideal Solenoid**—*Analytical Approach*

Equilibrium Equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + \frac{\partial \tau_{rz}}{\partial z} = -\lambda J B_z(r,z)$$

$$\frac{\partial \tau_{rz}}{\partial r} - \frac{\tau_{rz}}{r} + \frac{\partial \sigma_z}{\partial z} = -\lambda J B_r(r,z)$$

- Ideal: *infinite length* Solenoid: *Turns no longer acting independently*

Boundary Conditions

$$\sigma_r(r=a_1,z)=0; \quad \sigma_r(r=a_2,z)=0; \quad \sigma_z(r,z=\pm b)=0;$$

$$\tau_{rz}(r=a_1,z)=0; \quad \tau_{rz}(r=a_2,z)=0; \quad \tau_{rz}(r,z=\pm b)=0$$

Strains

$$\epsilon_r = \frac{1}{E_r}\sigma_r - \frac{\nu_{\theta r}}{E_\theta}\sigma_\theta - \frac{\nu_{zr}}{E_z}\sigma_z + \epsilon_{T_r}$$

$$\epsilon_\theta = -\frac{\nu_{r\theta}}{E_r}\sigma_r + \frac{1}{E_\theta}\sigma_\theta - \frac{\nu_{z\theta}}{E_z}\sigma_z + \epsilon_{T_\theta}$$

$$\epsilon_z = -\frac{\nu_{rz}}{E_r}\sigma_r - \frac{\nu_{\theta z}}{E_\theta}\sigma_\theta + \frac{1}{E_z}\sigma_z + \epsilon_{T_z}$$

$$\gamma_{rz} = \frac{1}{G_{rz}}\tau_{rz}$$

Strains

$$\epsilon_{T_r} = \int_{300\,\mathrm{K}}^{T_{op}} \alpha_{T_r}(T)\,dT;$$

$$\epsilon_{T_\theta} = \int_{300\,\mathrm{K}}^{T_{op}} \alpha_{T_\theta}(T)\,dT;$$

$$\epsilon_{T_z} = \int_{300\,\mathrm{K}}^{T_{op}} \alpha_{T_z}(T)\,dT$$

Radial and Hoop Stresses

$$\sigma_\rho = \frac{\lambda J B_1 a_1}{\alpha - 1}\left[\frac{2+\nu}{3}(\alpha-\kappa)\left(\frac{\alpha^2+\alpha+1-\alpha^2/\rho^2}{\alpha+1}-\rho\right) - \frac{3+\nu}{8}(1-\kappa)\left(\alpha^2+1-\frac{\alpha^2}{\rho^2}-\rho^2\right)\right]$$

$$\sigma_\theta = \frac{\lambda J B_1 a_1}{\alpha - 1}\left\{(\alpha-\kappa)\left[\frac{2+\nu}{3}\left(\frac{\alpha^2+\alpha+1+\alpha^2/\rho^2}{\alpha+1}\right) - \frac{1+2\nu}{3}\rho\right] - (1-\kappa)\left[\frac{3+\nu}{8}\left(\alpha^2+1+\frac{\alpha^2}{\rho^2}\right)-\frac{1+3\nu}{8}\rho^2\right]\right\}$$

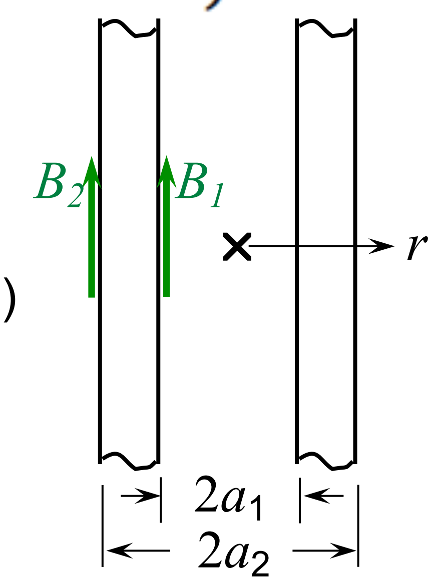
$\alpha = a_2/a_1$

$\rho = r/a_1$

$\kappa = B_2/B_1$*

(*$\kappa = 0$ for ∞ long)

$\nu \sim 0.3$



Hoop Stress, σ_θ, in Solenoid, vs. Radial Distance, $\rho = r/a_1$

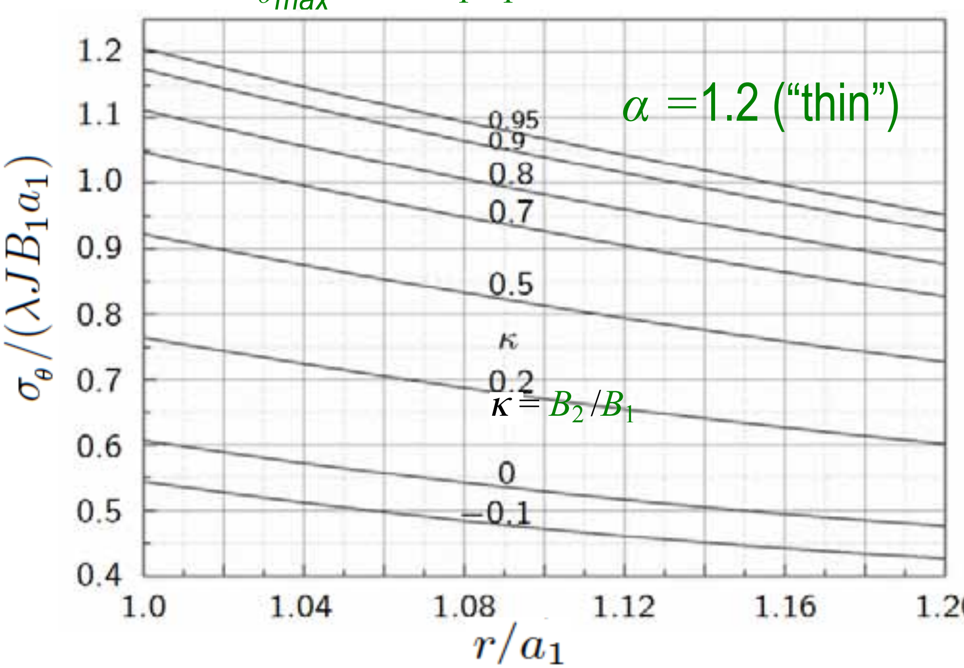
$$\sigma_\theta = \frac{\lambda J B_1 a_1}{\alpha - 1}\left\{(\alpha-\kappa)\left[\frac{2+\nu}{3}\left(\frac{\alpha^2+\alpha+1+\alpha^2/\rho^2}{\alpha+1}\right) - \frac{1+2\nu}{3}\rho\right] - (1-\kappa)\left[\frac{3+\nu}{8}\left(\alpha^2+1+\frac{\alpha^2}{\rho^2}\right) - \frac{1+3\nu}{8}\rho^2\right]\right\}$$

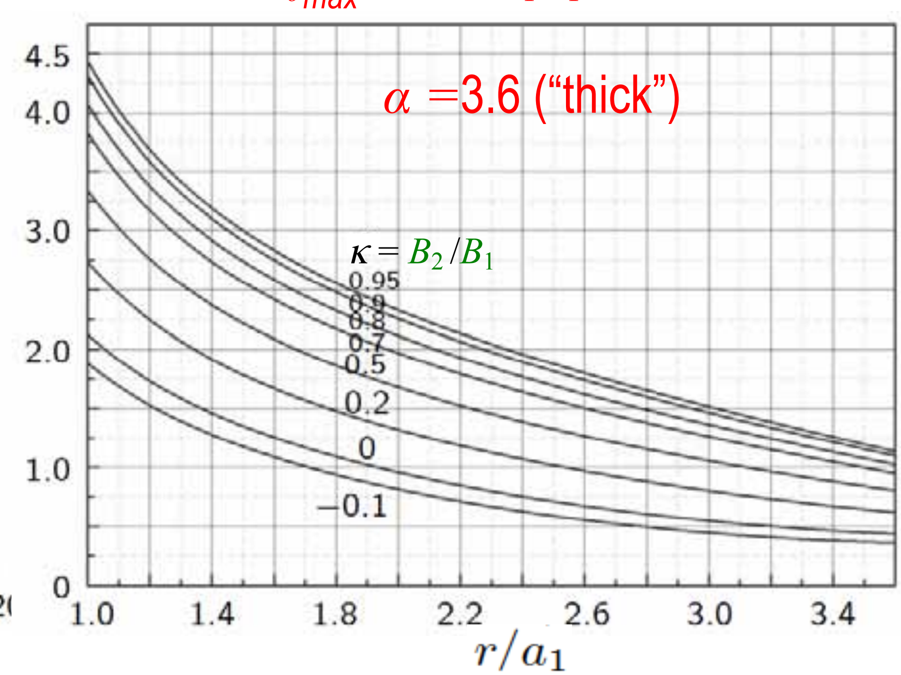
Thin Coil: $\lim_{\alpha\to 1} \sigma_\theta \to \frac{1}{2}(1+\kappa)\lambda J B_1 a_1$

Thick Coil: $\lim_{\alpha\gg 1} \sigma_\theta \to \alpha\left[\frac{(7+9\kappa)+(5+3\kappa)\nu}{12}\right]\lambda J B_1 a_1$

$\sigma_{\theta max} = \lambda J B_1 a_1$ for $\alpha = 1$ & $\kappa = 1$

$\sigma_{\theta max} >> \lambda J B_1 a_1$ for $\alpha >> 1$





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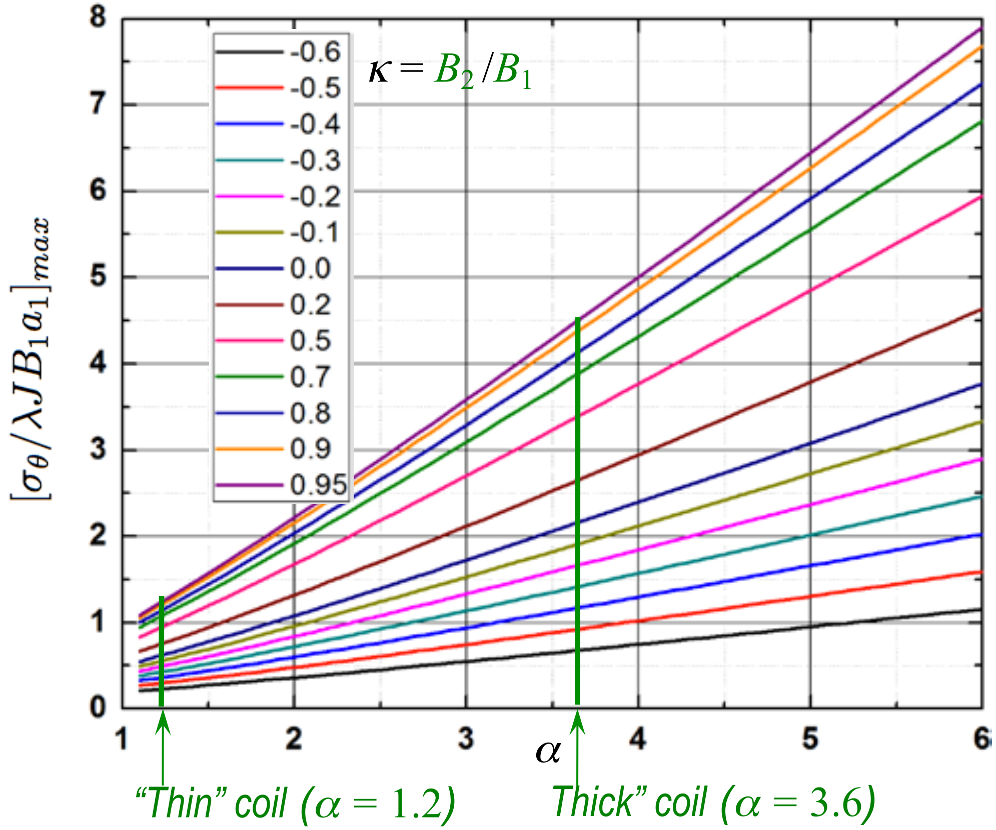
Maximum Hoop Stress, $\sigma_{\theta max}$, *at* $r = a_1$

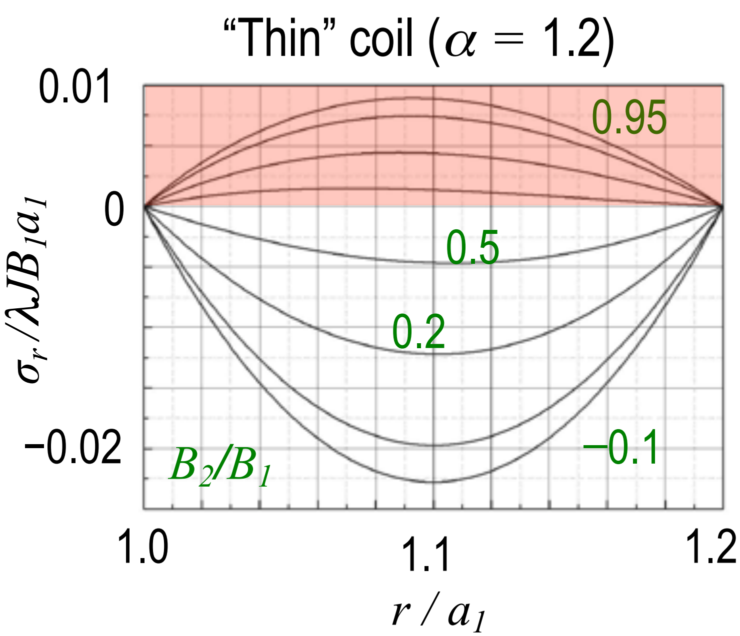


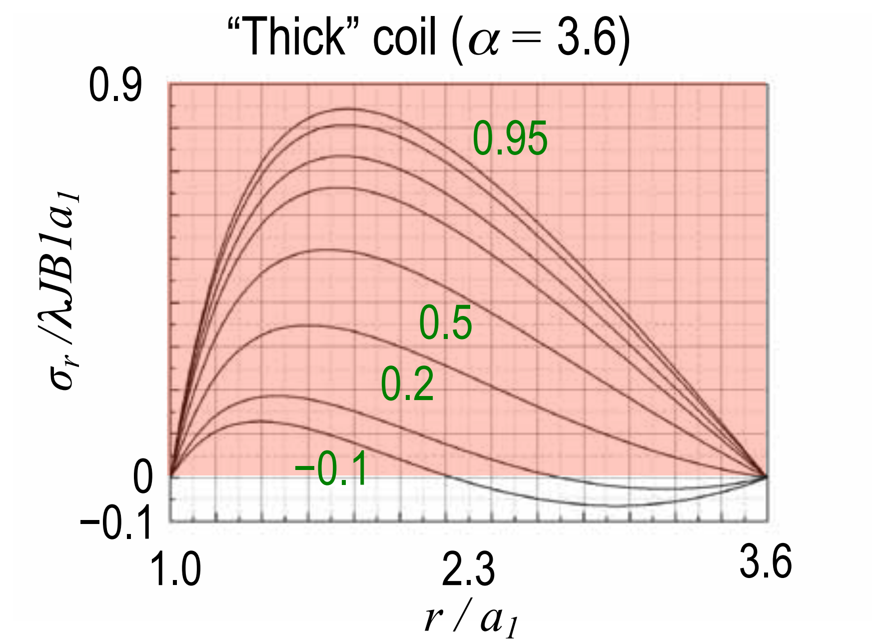
For high-field coil: "thin" radial built, i.e., $\alpha \rightarrow 1$

$$\sigma_{\theta max} \rightarrow \lambda J B_1 a_1$$

Radial Stress, $\sigma_r/\lambda J B_1 a_1$, vs. *Radial Distance*, $\rho = r/a_1$

$$\sigma_\rho = \frac{\lambda J B_1 a_1}{\alpha - 1}\left[\frac{2+\nu}{3}(\alpha-\kappa)\left(\frac{\alpha^2+\alpha+1-\alpha^2/\rho^2}{\alpha+1}-\rho\right) - \frac{3+\nu}{8}(1-\kappa)\left(\alpha^2+1-\frac{\alpha^2}{\rho^2}-\rho^2\right)\right]$$

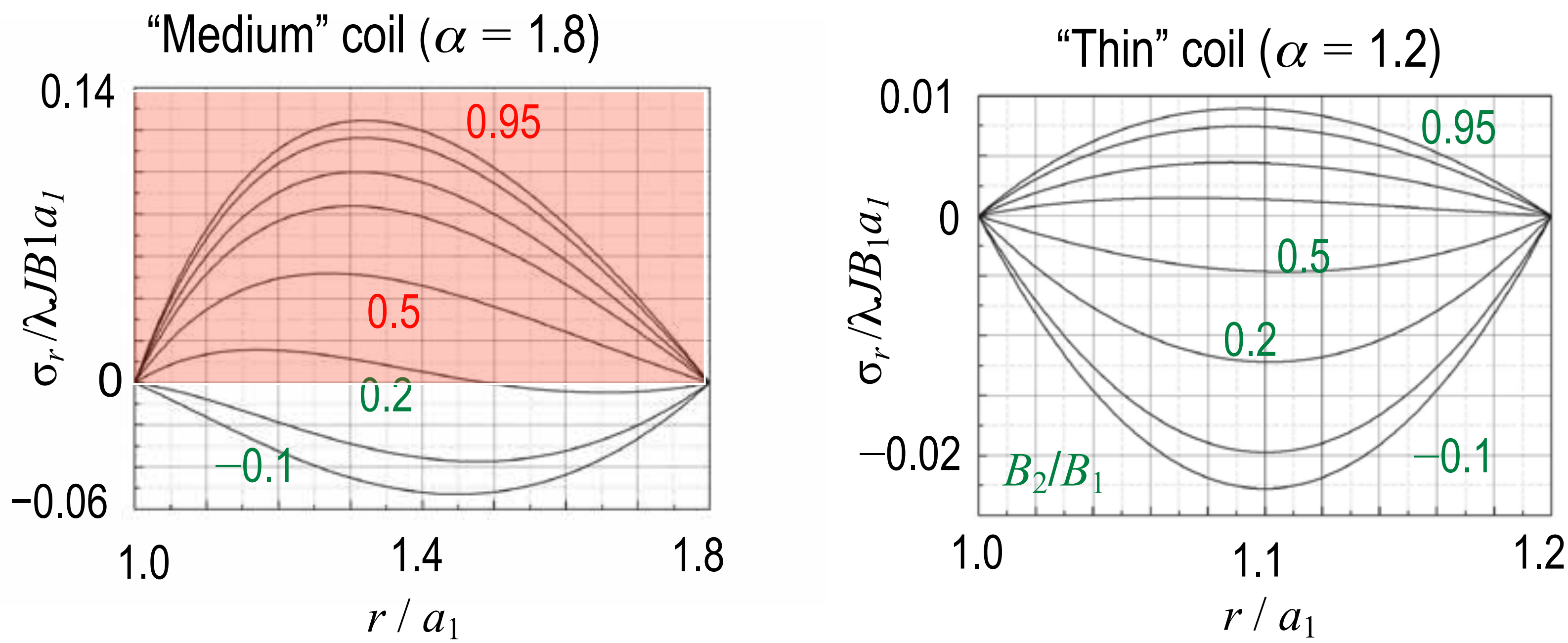




- $\sigma_r < 0$ to keep the winding from separating, Coil, "thin" radial built, i.e., $\alpha \rightarrow 1$

Radial Stress ($\sigma_r/\lambda J B_1 a_1$) vs. Radial Distance ($r/a_1$)

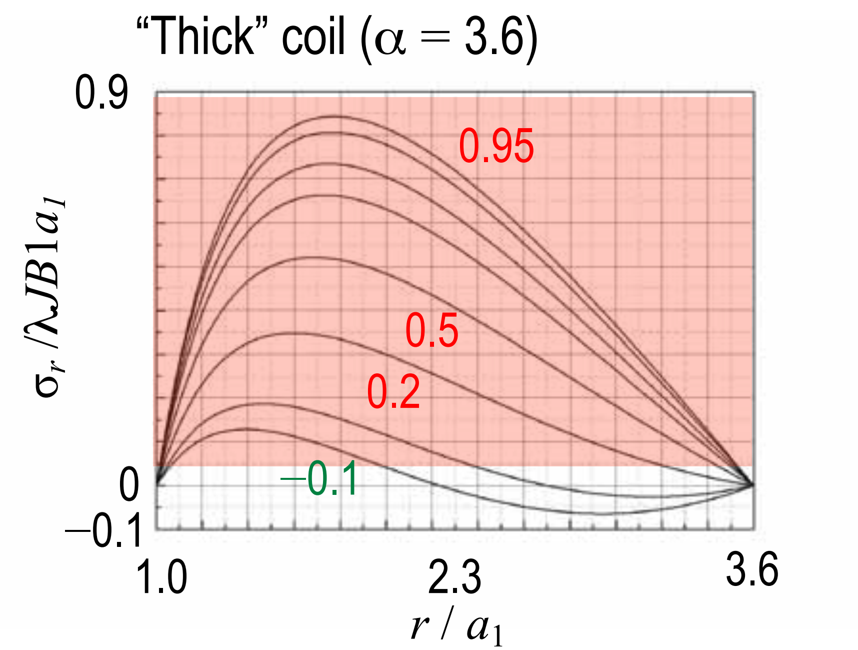
"Medium" Coil (α = 1.8)

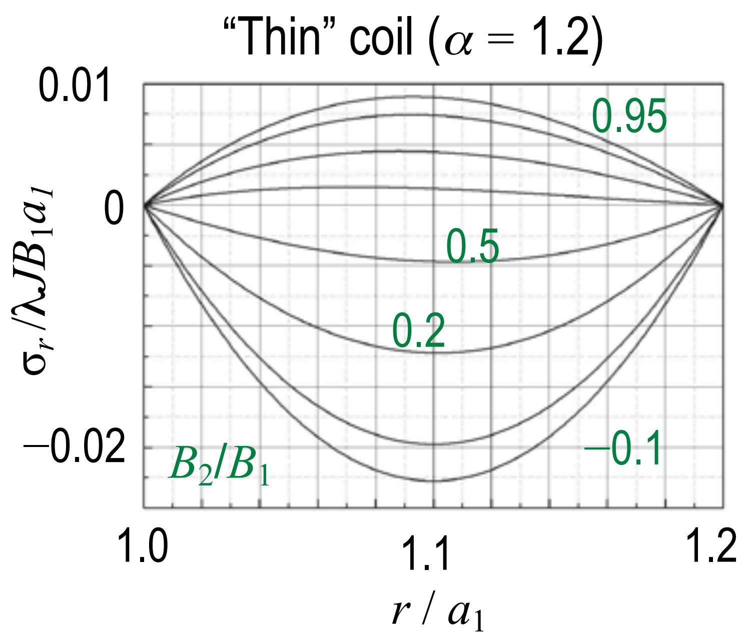


- Desirable to split this coil into two "thin-walled" coils

Radial Stress ($\sigma_r/\lambda J B_1 a_1$) vs. Radial Distance ($r/a_1$)

"Thick" Coil (α = 3.6)





An Illustration of a "Thick, Thick" Coil: a 100-T Magnet Design

$$B_\circ = \mu_\circ \lambda J a_1 F(\alpha, \beta)$$

$$F(\alpha,\beta) = \beta \ln\left(\frac{\alpha + \sqrt{\alpha^2+\beta^2}}{1+\sqrt{1+\beta^2}}\right)$$

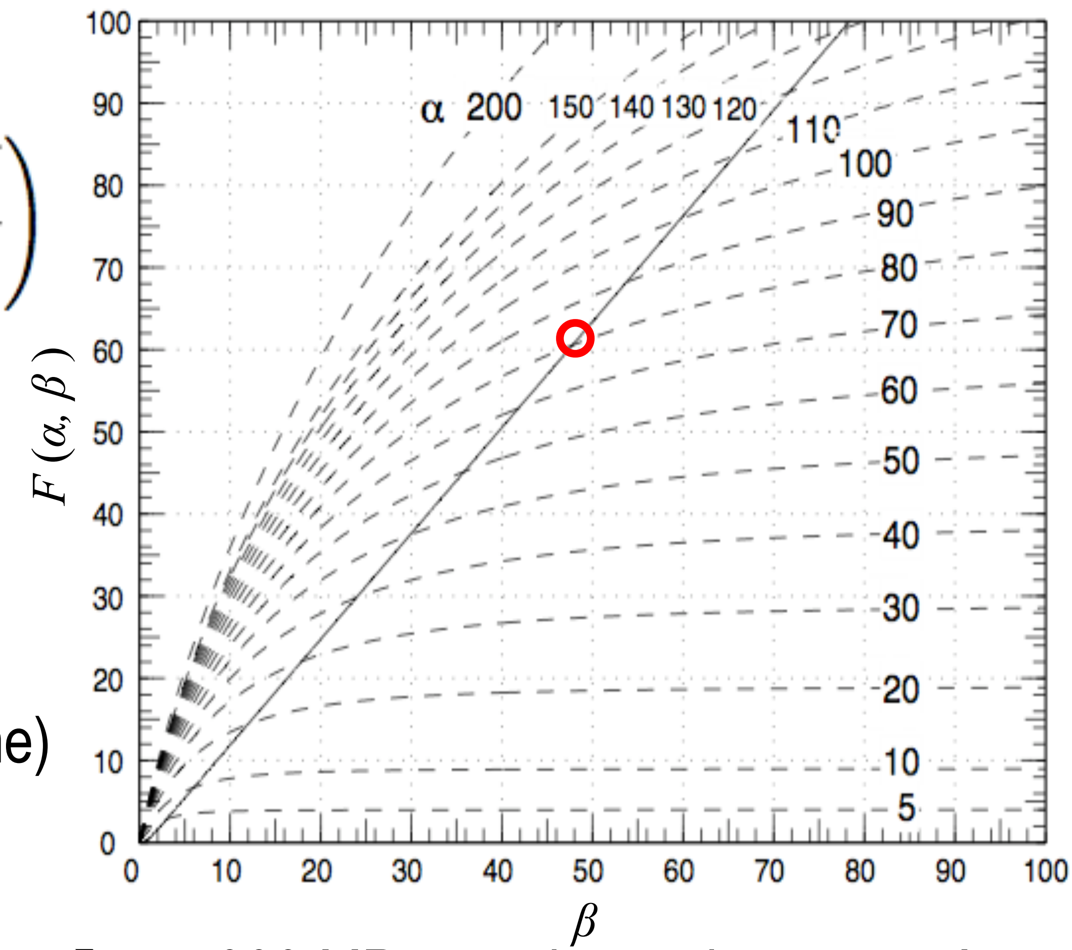
$2a_1 = 10$ mm (winding i.d.)

$\lambda J = 270 \times 10^6$ A/m^2

$B_o = 100$ T ➔ $F(\alpha, \beta) \sim 60$

$\alpha = 80$; $\beta = 47$ (minimum volume)

$2a_2 = 800$ mm; $2b = 470$ mm



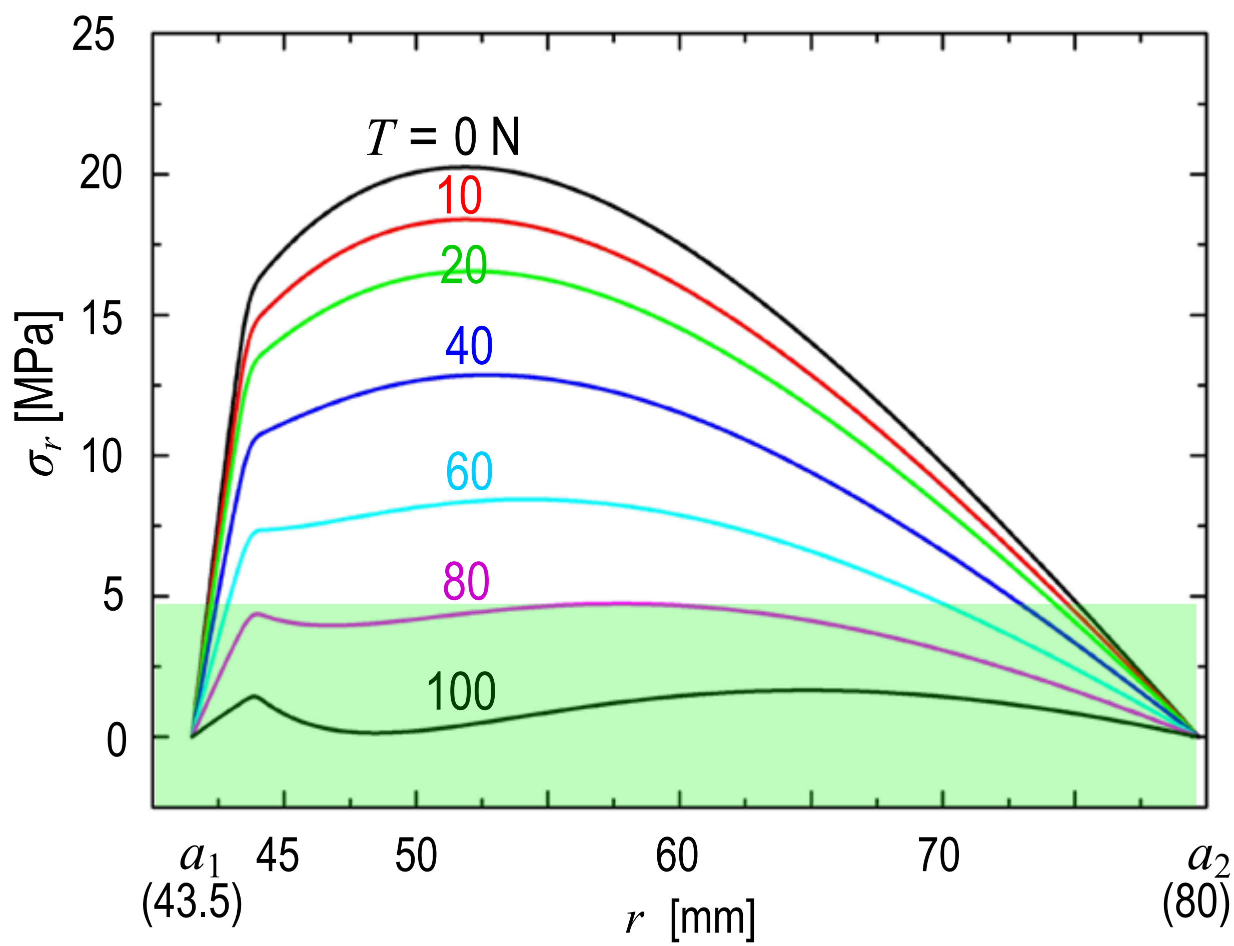
For $\alpha \approx 80$ ➔ $\sigma_{\theta max} \approx 51 \lambda J B_1 a_1$ ➔ *~7000 MPa*, much, much too great!

- A feasible 100-T magnet consists of 39 nested thin coils, each $\alpha \approx 1$*
  - Yukikazu Iwasa and Seungyong Hahn "First-cut design of an all-superconducting 100-T direct current magnet," *App. Phys. Lett.* **103,** 253507 (2013).

Approaches to keep $\sigma_r < 0$ in the Winding

- Keep the winding "thin"
- A "thick" coil is often split into 2, 3, 4… "thin" coils:
  - To keep $\sigma_r < 0$
  - Equally importantly, for conductor "grading"
- Use a "bladder" (e.g., *ATLAS*) or "over-banding"
  - Effective only for "thin" winding
- Winding tension

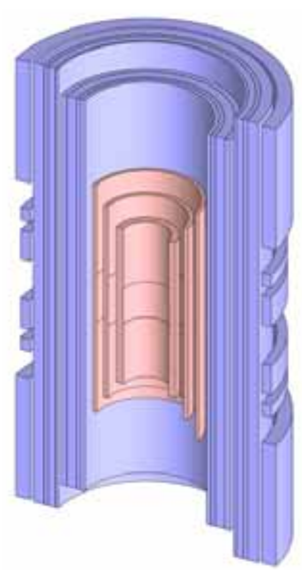
Effects of Winding Tension, e.g. ($\alpha = 1.84$)



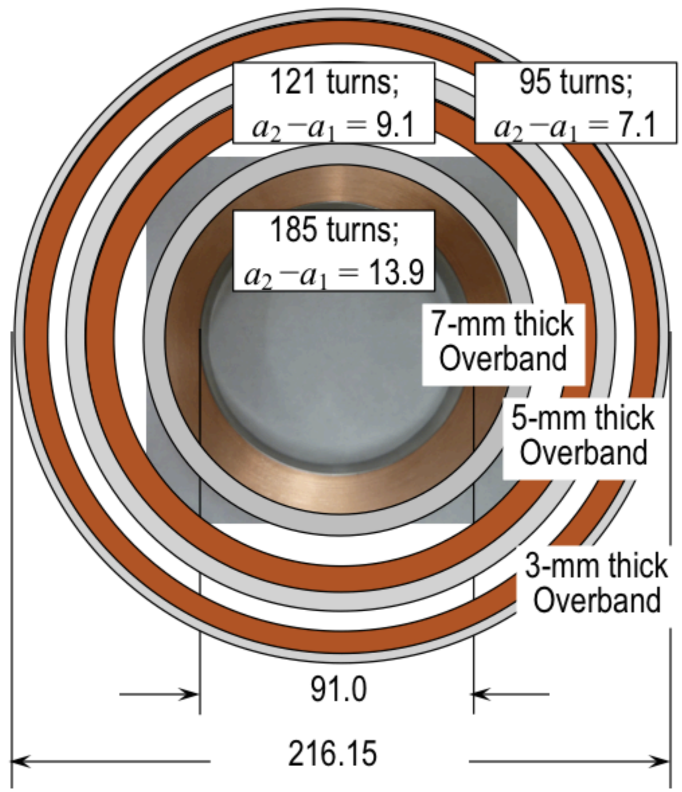
Overbanding

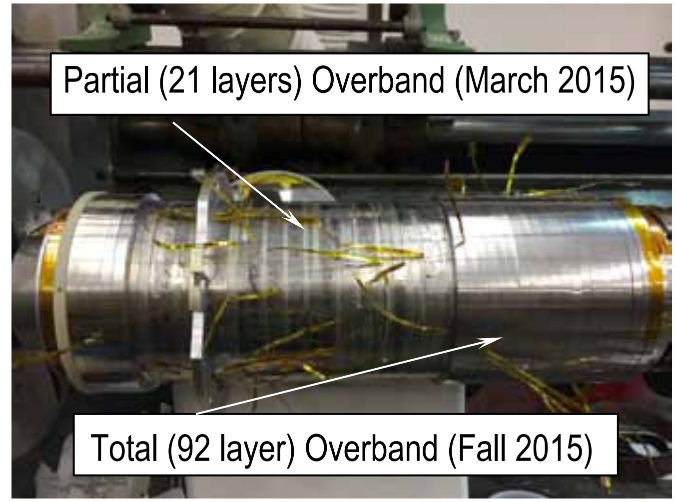
Overbanding of H800

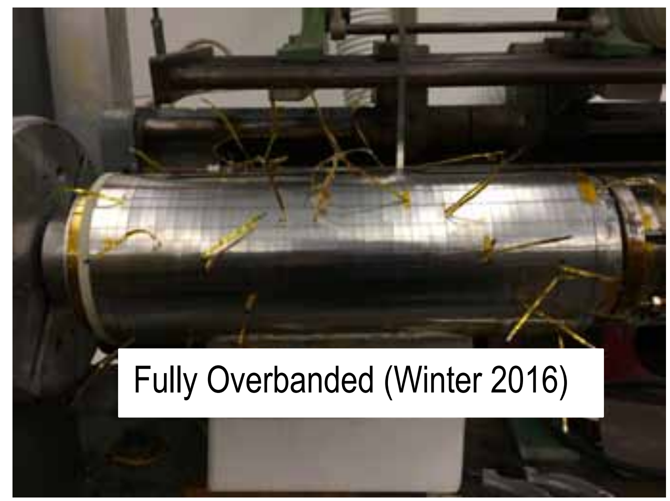
MIT 1.3-GHz LTS/HTS
NMR Magnet
(L500/H800)



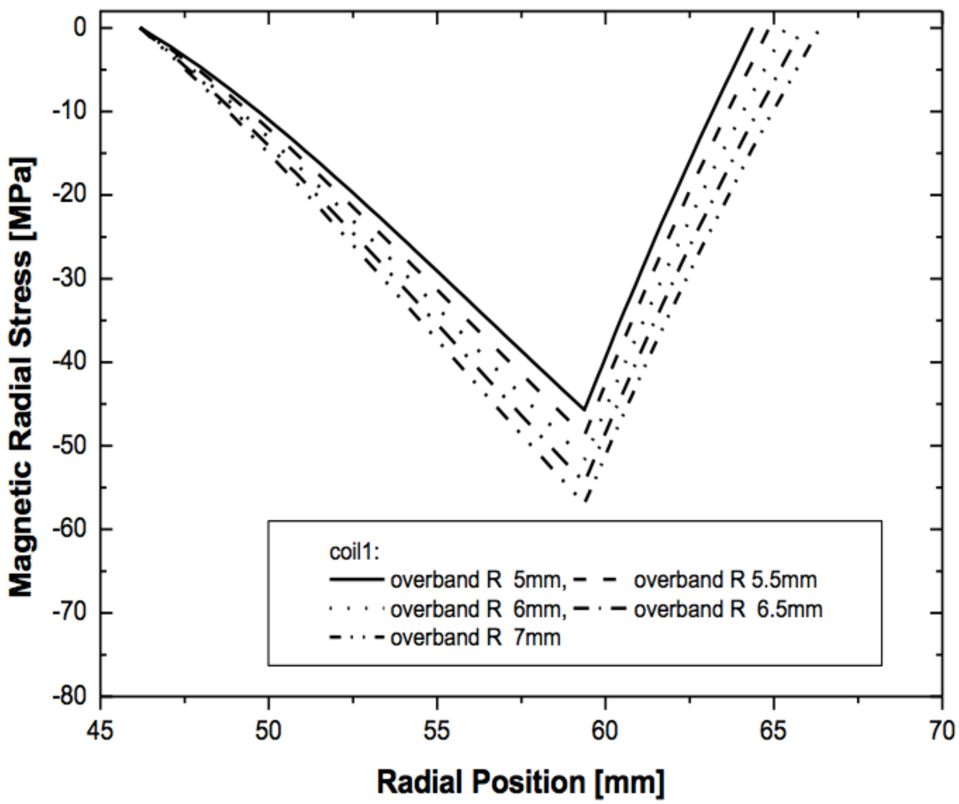
H800 (18.79 T): I_{op}=251 A
Coil 1 (6-mm REBCO: 26 NI-DP)
Coil 2 (6-mm REBCO: 32 NI-DP)
Coil 3 (6-mm REBCO: 38 NI-DP)

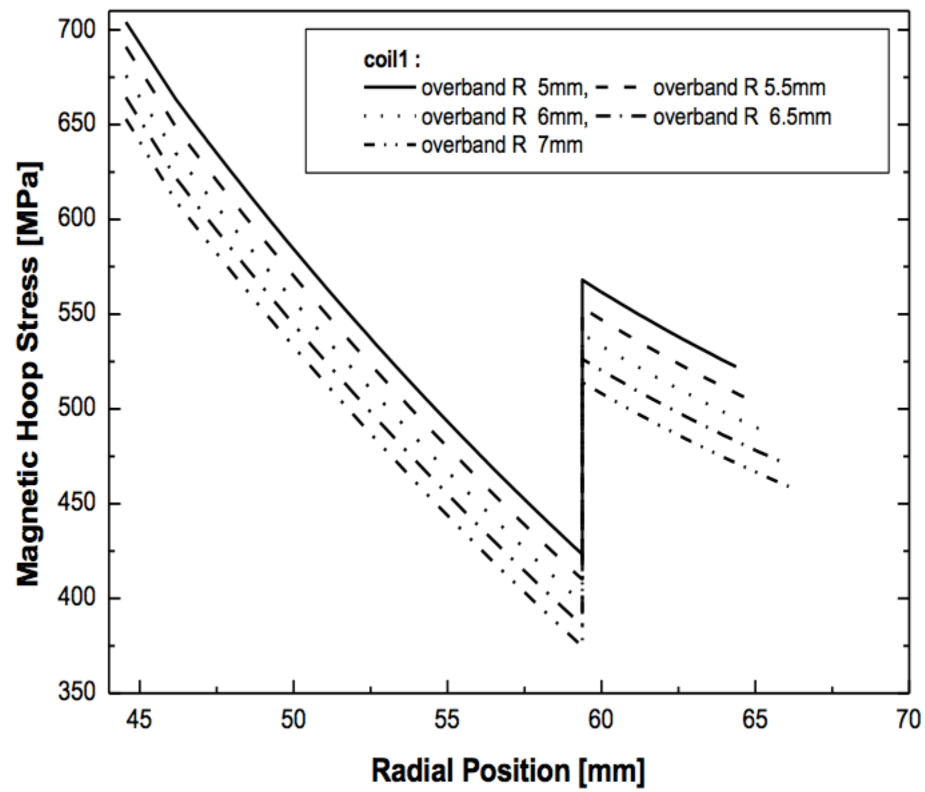






Overbanding H800 Coils*





- Mingzhi Guan, Seungyong Hahn, Juan Bascuñán, Timing Qu, Xingzhe Wang, Peifeng Gao, and Yukikazu Iwasa, "A parametric study on overband radial build for a REBCO 800-MHz Insert of a 1.3-GHz LTS/HTS NMR magnet." presented at MT24.

Conclusions on Stresses

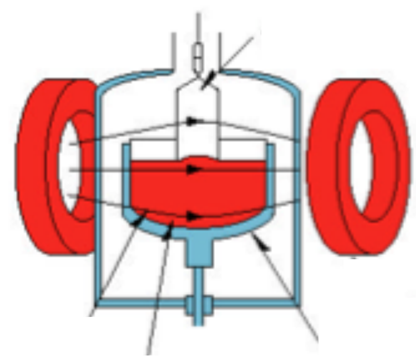
- σ_r is beneficial in reducing σ_θ in solenoids *if* it is *compressive*
- σ_r makes matters worse if it is *tensile*
- Tensile stress bad for insulation—film and epoxy resins cannot take much tension before cracking or separating
  - Could cause winding delamination
  - Could lead to energy release and quenching
- Often pre-stress is applied at RT during coil fabrication to maintain only radial compression under all conditions of cool-down and operation

Other Considerations

- For thick windings, divide the coil into several thinner, mechanically separate, concentric sections to keep $\sigma_r < 0$
- The assumption of isotropic properties (elastic) is often invalid: $E_{metal} >> E_{insulation}$, making windings "spongy" in the radial direction

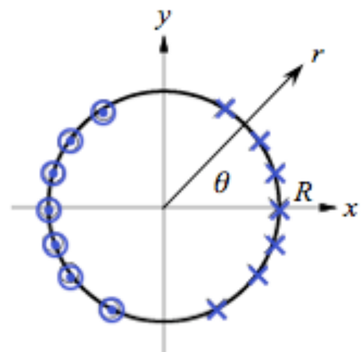
Axial forces always cause compressive stresses; no coupling with σ_r and σ_θ

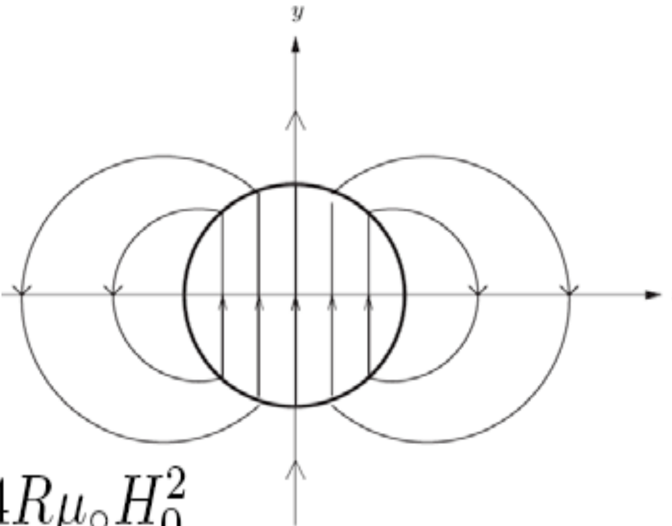
- Compute independently and sum them up
- Insulating materials often strong (or at least adequate) in compression
- Special case: a *split pair coil* requires extra structure to bridge the gap



Forces in Ideal Dipole

$$\vec{f}_L = \vec{K}_f \times \mu_\circ H_0 \sin\theta \, \vec{\imath}_r$$
$$= -2\mu_\circ H_0^2 \cos\theta \, \sin\theta \, \vec{\imath}_\theta$$
$$= -\mu_\circ H_0^2 \sin 2\theta \, \vec{\imath}_\theta \quad \text{[N/m}^2\text{]}$$

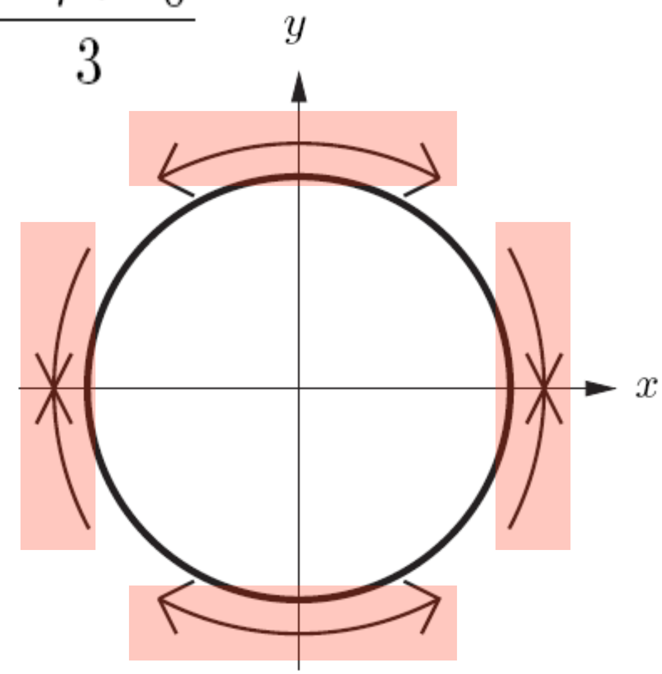




- Horizontal: pushing out

$$F_{Ldx} = \int \vec{f}_L \cdot \vec{\imath}_x \, dx = -R \int_{-\pi/2}^{\pi/2} f_{L\theta} \sin\theta \, d\theta = \frac{4R\mu_\circ H_0^2}{3}$$

  - Steel collar & yoke
- Vertical: crushing
  - More difficult force to handle; chief cause of premature quenches
- Both forces cause compressive stresses in the winding



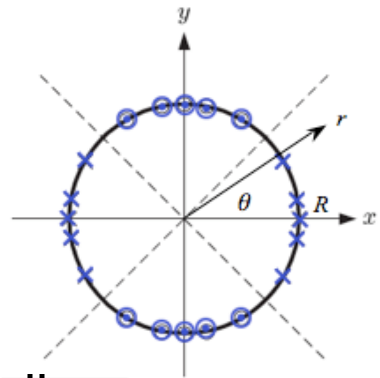
Forces in Ideal Quadrupole

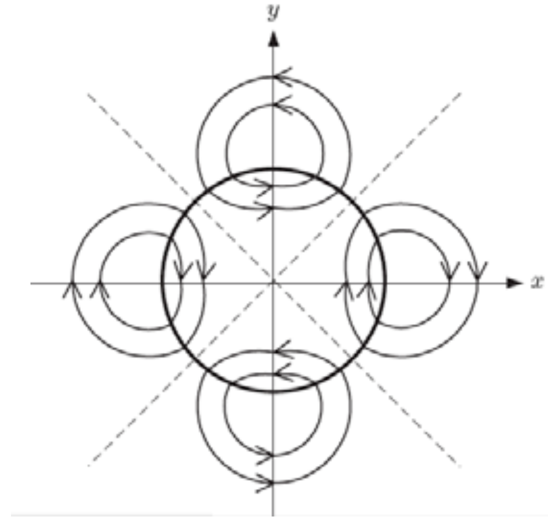
$$\vec{f}_L = -2\mu_\circ H_0^2 \sin 2\theta \cos 2\theta \, \vec{\imath}_\theta$$

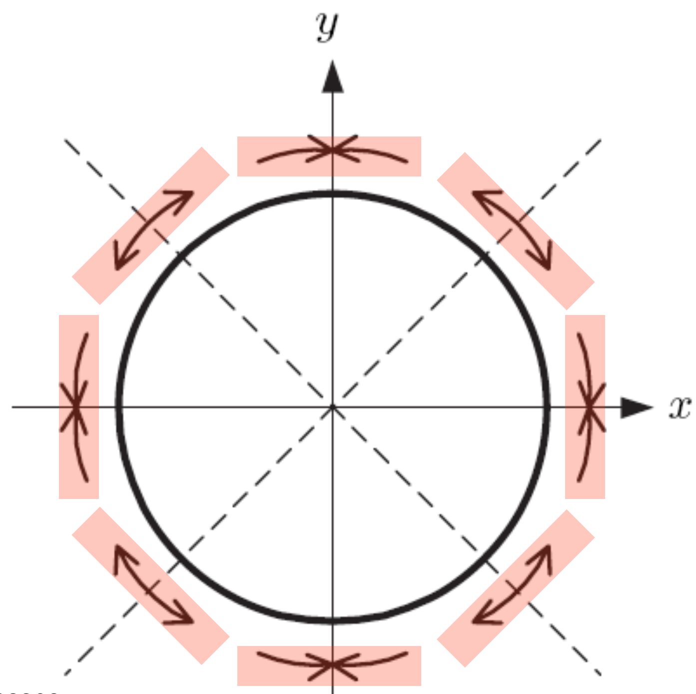
$$= -\mu_\circ H_0^2 \sin 4\theta \, \vec{\imath}_\theta \quad \text{[N/m}^2\text{]}$$

Same as dipole's

- Crushing, horizontally & vertically
- Pushing out along ± 45° axes







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Axial Forces

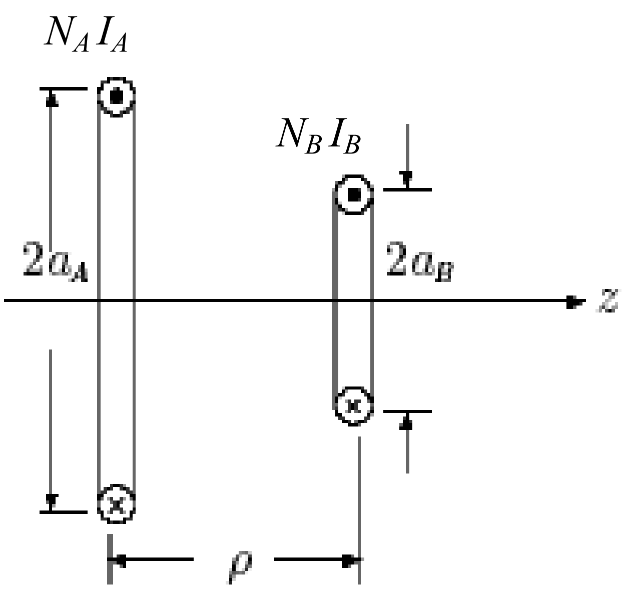
1. Axial Force Between Two "Ring" Coils

$$F_{zA}(\rho) = \frac{\mu_\circ}{2}(N_A I_A)(N_B I_B)\frac{\rho\sqrt{(a_A + a_B)^2 + \rho^2}}{(a_A - a_B)^2 + \rho^2}$$

$$\times\left\{k^2 K(k) + (k^2 - 2)[K(k) - E(k)]\right\}$$

$K(k)$ and $E(k)$ the complete Elliptic Integrals, respectively, of the 1st and 2nd kinds

$$k^2 = \frac{4a_A a_B}{(a_A + a_B)^2 + \rho^2}$$



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Complete Elliptic Integrals of the 1st and 2nd Kinds

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta$$

k^2	k	$K(k)$	$E(k)$	k^2	k	$K(k)$	$E(k)$
0	0	$\pi/2$	$\pi/2$	0.7	0.8367	2.0754	1.2417
0.1	0.3162	1.6124	1.5308	0.8	0.8944	2.2572	1.1785
0.2	0.4472	1.6596	1.4890	0.90	0.9487	2.5781	1.1048
0.3	0.5477	1.7139	1.4454	0.95	0.9747	2.9083	1.0605
0.4	0.6325	1.7775	1.3994	0.98	0.9899	3.3541	1.0286
0.5	0.7071	1.8541	1.3506	0.99	0.9950	3.6956	1.0160
0.6	0.7746	1.9496	1.2984	1	1	∞	1

$K(k)$ & $E(k)$

$$K(k) = \frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2 k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6+\left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2 k^8+\cdots\right]$$

$$E(k) = \frac{\pi}{2}\left[1-\left(\frac{1}{2}\right)^2 k^2-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5}-\left(\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\right)^2 \frac{k^8}{7}-\cdots\right]$$

$$K(k) \simeq \frac{\pi}{2}\left(1+\tfrac{1}{4}k^2+\tfrac{9}{64}k^4+\tfrac{25}{256}k^6+\tfrac{1225}{16384}k^8\right)$$

$$E(k) \simeq \frac{\pi}{2}\left(1-\tfrac{1}{4}k^2-\tfrac{3}{64}k^4-\tfrac{5}{256}k^6-\tfrac{175}{16384}k^8\right)$$

$$K(k)-E(k) \simeq \frac{\pi}{4}\left(k^2+\tfrac{3}{8}k^4+\tfrac{15}{64}k^6+\tfrac{175}{1024}k^8\right)$$

Axial Force Between Two "Ring" Coils

In the limit $\rho^2 >> (a_A + a_B)^2 \;\Rightarrow\; k^2 \rightarrow 0$, i.e. two rings far apart:

$$F_{zA}(\rho) = \frac{\mu_\circ}{2}(N_A I_A)(N_B I_B)\boxed{\frac{\rho\sqrt{(a_A + a_B)^2 + \rho^2}}{(a_A - a_B)^2 + \rho^2}} \longrightarrow 1$$

$$\times \left\{k^2 K(k) + (k^2 - 2)[K(k) - E(k)]\right\}$$

$$\Downarrow$$

$$F_{zA}(\rho) \simeq \frac{\mu_\circ}{2}(N_A I_A)(N_B I_B)\left\{k^2 \underbrace{K(k)} + (k^2 - 2)\underbrace{[K(k) - E(k)]}\right\}$$

$$F_{zA}(\rho) \simeq \frac{\mu_\circ}{2}(N_A I_A)(N_B I_B)$$

$$\times \left[k^2 \overbrace{\left(\frac{\pi}{2} + \frac{\pi}{8}k^2\right)} + (k^2 - 2)\overbrace{\left(\frac{\pi}{4}k^2 + \frac{3\pi}{32}k^4\right)}\right]$$

$$\simeq \frac{\mu_\circ}{2}(N_A I_A)(N_B I_B)\left(\frac{3\pi}{16}k^4\right)$$

Axial Force Between Two "Ring" Coils

In the limit $\rho^2 >> (a_A + a_B)^2 \Rightarrow k^2 \rightarrow 0$, i.e. two rings far apart:

$$F_{zA}(\rho) = \frac{3\mu_\circ}{2\pi}\left(\frac{\pi a_A^2 N_A I_A}{\rho^2}\right)\left(\frac{\pi a_B^2 N_B I_B}{\rho^2}\right)$$

- $F_{ZA}(\rho) \propto$ to the product of each ring's "magnetic moment" but *each* reduced by $\boldsymbol{\rho^2}$
- Note that if both rings carry current in the same direction, $F_{ZA}(\rho) > 0$: *Attractive*

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Axial Force Within a "Thin-Walled" Solenoid

$$F_z(z) = -\frac{\mu_\circ}{2}\left(\frac{NI}{2b}\right)^2 \Big\{(b-z)\sqrt{4a^2+(b-z)^2}\left[K(k_{b_-})-E(k_{b_-})\right]$$
$$+(b+z)\sqrt{4a^2+(b+z)^2}\left[K(k_{b_+})-E(k_{b_+})\right]$$
$$-2b\sqrt{4a^2+4b^2}\left[K(k_{2b})-E(k_{2b})\right]\Big\}$$

$$k_{b_-}^2 = \frac{4a^2}{4a^2+(b-z)^2}; \qquad k_{b_+}^2 = \frac{4a^2}{4a^2+(b+z)^2}; \qquad k_{2b}^2 = \frac{4a^2}{4a^2+(2b)^2}$$

End Force: Zero because at $z = b$, $k_{b+} = k_{2b}$

This is expected

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Axial Force Within a "Thin-Walled" Solenoid

Midplane ($z = 0$) Force

$$F_z(0) = -\frac{\mu_o}{2}\left(\frac{NI}{2b}\right)^2\Big\{2b\sqrt{4a^2+b^2}\,[K(k_b)-E(k_b)]$$
$$-2b\sqrt{4a^2+4b^2}\,[K(k_{2b})-E(k_{2b})]\Big\}$$

$$k_b^2 = \frac{4a^2}{4a^2+b^2} \qquad k_{2b}^2 = \frac{4a^2}{4a^2+(2b)^2}$$

Axial Force Within a "Thin-Walled" Solenoid

Midplane ($z = 0$) Force in a "Long" Solenoid ($\beta >> 1$ or $k^2 << 1$)

$$k_{b_-}^2 = \frac{4a^2}{4a^2 + (b - z)^2}; \qquad k_{b_+}^2 = \frac{4a^2}{4a^2 + (b + z)^2}; \qquad k_{2b}^2 = \frac{4a^2}{4a^2 + (2b)^2}$$

$$k_{b_-}^2 = k_{b_+}^2 = k_b^2 \Longrightarrow \frac{4a^2}{b^2}; \qquad k_{2b}^2 \Longrightarrow \frac{4a^2}{4b^2}$$

two terms identical ($z = 0$)

$$F_z(z) = -\frac{\mu_o}{2}\left(\frac{NI}{2b}\right)^2 \Big\{ (b - z)\sqrt{4a^2 + (b - z)^2}\,[K(k_{b_-}) - E(k_{b_-})] + (b + z)\sqrt{4a^2 + (b + z)^2}\,[K(k_{b_+}) - E(k_{b_+})] - 2b\sqrt{4a^2 + 4b^2}\,[K(k_{2b}) - E(k_{2b})] \Big\}$$

$$F_z(0) = -\frac{\mu_o}{2}\left(\frac{NI}{2b}\right)^2 \Big\{ 2b\sqrt{4a^2 + b^2}\,[K(k_b) - E(k_b)] - 2b\sqrt{4a^2 + 4b^2}\,[K(k_{2b}) - E(k_{2b})] \Big\}$$

Midplane Force in a "Long" Solenoid ($\beta >> 1$ or $k^2 << 1$)

$$F_z(0) = -\frac{\mu_\circ}{2}\left(\frac{NI}{2b}\right)^2\Big\{2b\sqrt{4a^2+b^2}\,[K(k_b)-E(k_b)]$$

$$-2b\sqrt{4a^2+4b^2}\,[K(k_{2b})-E(k_{2b})]\Big\}$$

$$K(k_b) - E(k_b) \Longrightarrow \frac{\pi}{4}k_b^2 = \frac{\pi a^2}{b^2}$$

$$K(k_{2b}) - E(k_{2b}) \Longrightarrow \frac{\pi}{4}k_{2b}^2 = \frac{\pi a^2}{4b^2}$$

$$F_z(0) \simeq -\frac{\mu_\circ}{2}\left(\frac{NI}{2b}\right)^2 \pi a^2 \;\Rightarrow\; F_z(0) \simeq -\tfrac{1}{2}\mu_\circ H_z^2(0,0)\times \pi a^2$$

Midplane force: ~ magnetic pressure *times* bore area

An Illustration

"Ballpark" midplane force, $F_z(0)$:

$$F_z(0) \simeq -\tfrac{1}{2}\mu_o H_z^2(0,0) \times \pi a^2$$

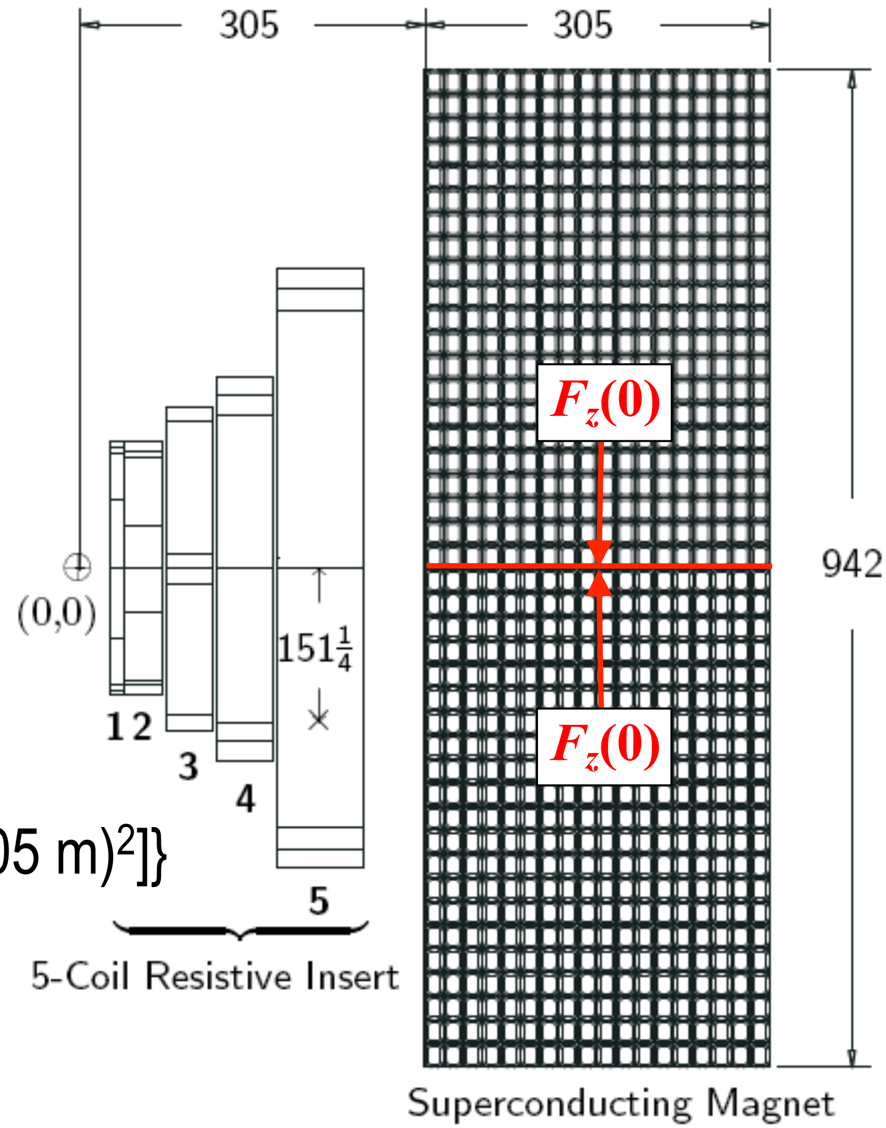
$\mu_o H_z(0,0) \cong 14$ T

$F_z(0) \cong -0.5[(14\ \mathrm{T})^2/(4\pi \times 10^{-7}\ \mathrm{H/m})]$
$\times(\pi)(0.305\ \mathrm{m})^2 = 23$ MN

"Ballpark" midplane compressive stress:

$\sigma_z(0) = F_z(0)/[\pi(a_2^2 - a_1^2)]$
$= (23\ \mathrm{MN})/\{\pi[(0.610\ \mathrm{m})^2 - (0.305\ \mathrm{m})^2]\}$

$\cong 26$ MPa



Conclusions

"Sometimes there is as much magic as science
in the explanations of the force.
Yet what is a magician but a practicing theorist?"

—Obi Wan Kenobi

I hope this lecture has given you a basic understanding of force and analytical tools in dealing with forces, stresses, and strains

Bonnes vacances sous le soleil
Reposez vous bien!