Covariant extension of Generalized Parton Distributions

From an Overlap of Light-cone Wave-functions to a Double Distribution

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Outline

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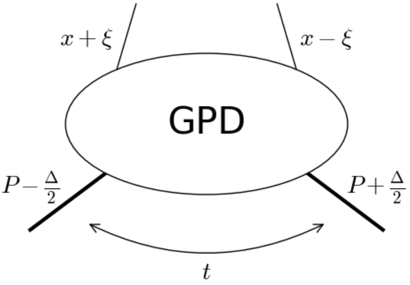
Definition of GPDs

- Quark GPD (twist-2, spin-0 hadron): (Müller et al., 1994; Radyushkin, 1996; Ji, 1997)

$$H^q(x,\xi,t) = \frac{1}{2}\int \frac{\mathrm{d}z^-}{2\pi}\, e^{i\,x\,P^+ z^-} \left\langle P+\frac{\Delta}{2}\left|\bar{q}(-z)\,\gamma^+ q(z)\right| P-\frac{\Delta}{2}\right\rangle\Bigg|_{z^+=0,\, z_\perp=0} . \tag{1}$$



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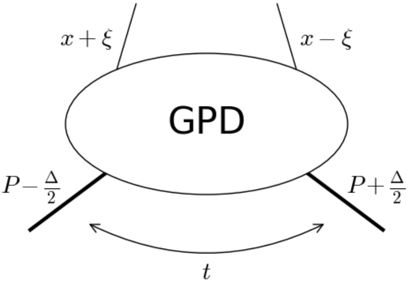

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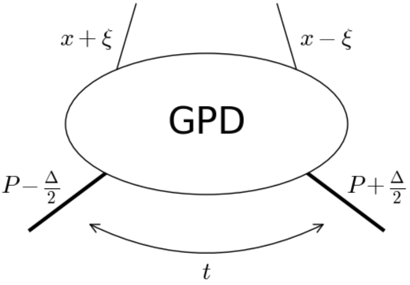

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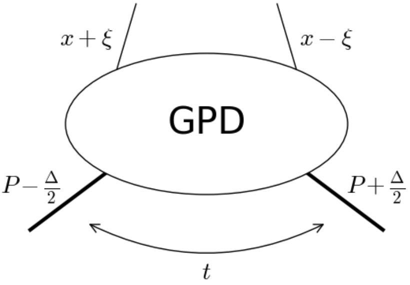

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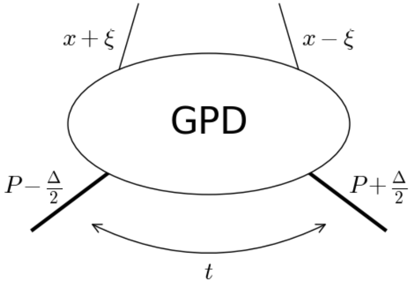

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- Impact parameter space GPD (at $\xi = 0$): (Burkardt, 2000)

$$q\left(x,\vec{b_\perp}\right) = \int \frac{\mathrm{d}^2\vec{\Delta_\perp}}{(2\pi)^2}\, e^{-i\,\vec{b_\perp}\cdot\vec{\Delta_\perp}}\, H^q\left(x,0,-\vec{\Delta_\perp}^{\,2}\right) . \tag{2}$$

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  - Cauchy-Schwarz theorem in Hilbert space.

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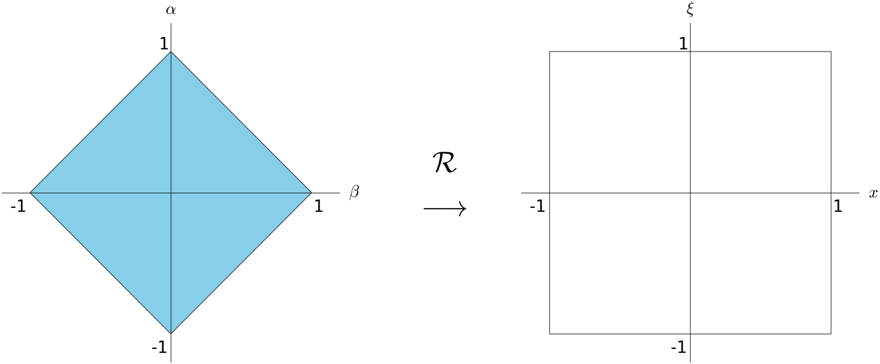
- Pobylitsa gauge (One Component DD): **(Pobylitsa, 2003)**

$$H(x,\xi,t) = (1-x)\int_\Omega \mathrm{d}\beta\,\mathrm{d}\alpha\, f(\beta,\alpha,t)\,\delta(x-\beta-\alpha\xi)\ , \quad (12)$$

with

$$\begin{cases} F(\beta,\alpha) & = & (1-\beta)\ f(\beta,\alpha) \\ G(\beta,\alpha) & = & -\alpha\, f(\beta,\alpha) \end{cases} . \quad (13)$$

Radon transform



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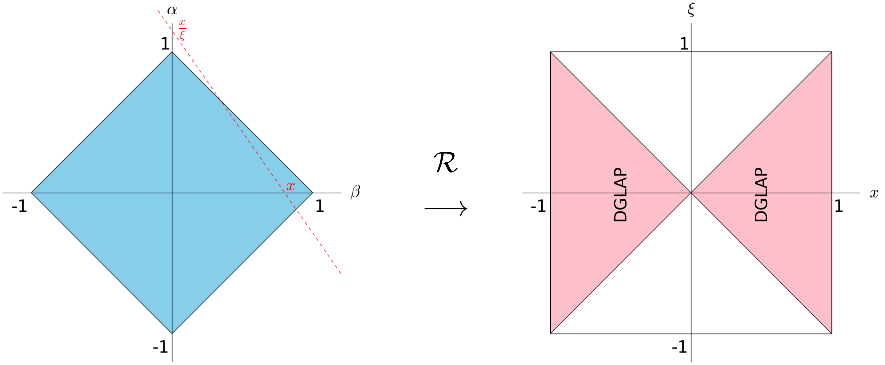

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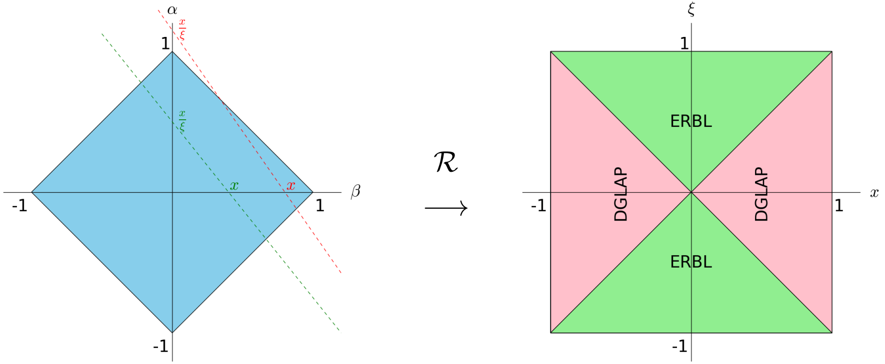
Radon transform



- Radon Transform:

$$\mathcal{R}f\,(x,\xi) \propto \int \mathrm{d}\beta\,\mathrm{d}\alpha\, f\,(\beta,\alpha)\;\delta\,(x-\beta-\alpha\xi)\;. \qquad (14)$$

  - DGLAP region: $|x| > |\xi|$.
  - ERBL region: $|x| < |\xi|$.

Outline

From DGLAP GPD to a DD

- In Overlap representation: DGLAP region only (e.g. two-body LCWFs).

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Problem

Find $f(\beta, \alpha)$ on square $\{|\alpha| + |\beta| \leq 1\}$ such that

$$H(x, \xi)|_{\mathrm{DGLAP}} = (1 - x) \int \mathrm{d}\beta\, \mathrm{d}\alpha\, f(\beta, \alpha)\, \delta(x - \beta - \alpha\xi) \ .$$

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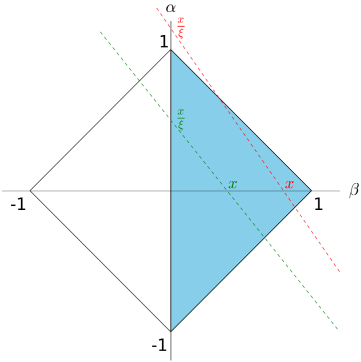
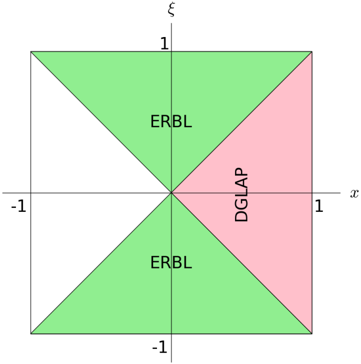
- If model fulfills Lorentz invariance: (Moutarde, 2015)
  - DD $f(\beta, \alpha)$ **exists** (if the GPD behaves well) and is **unique**.
  - We can reconstruct the GPD everywhere.

Support properties



$\mathcal{R}$
$\longrightarrow$



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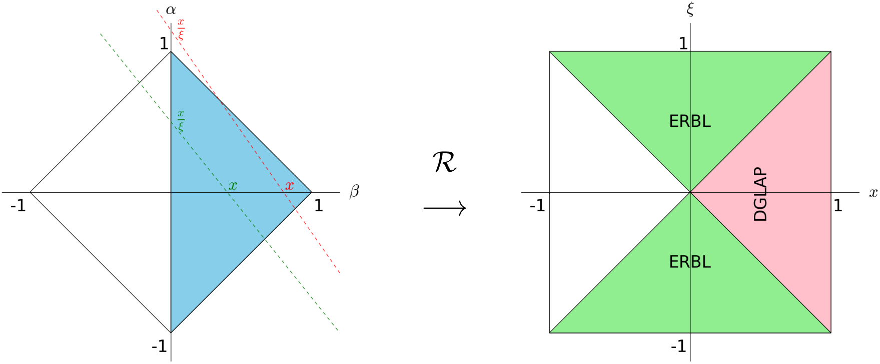
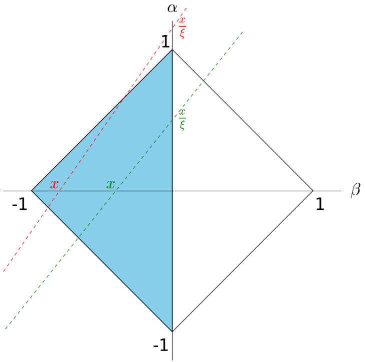
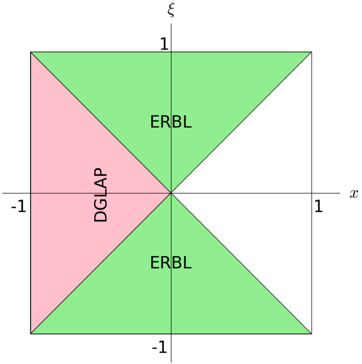


- Quark GPD: $H(x,\xi) = 0$ for $-1 < x < -|\xi| \Longrightarrow f(\beta,\alpha) = 0$ for $\beta < 0$.

Support properties



$$\xrightarrow{\mathcal{R}}$$



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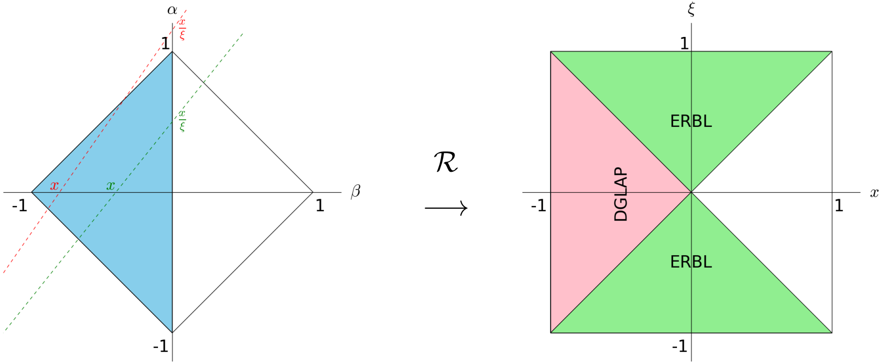

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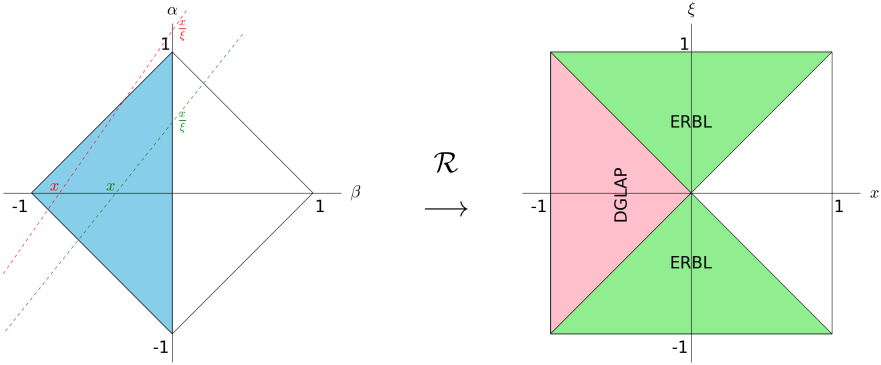

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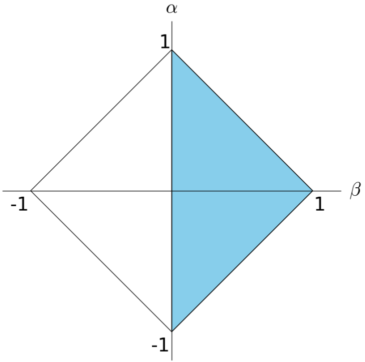
- Domains $\beta < 0$ and $\beta > 0$ are uncorrelated in the DGLAP region.
- Divide and conquer:
  - Better numerical stability.
  - Lesser complexity: $O(N^p + N^p) \ll O\left((N+N)^p\right)$.

Domain for the inversion



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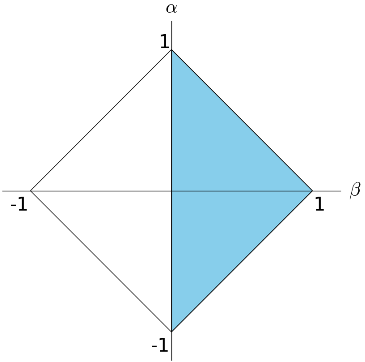

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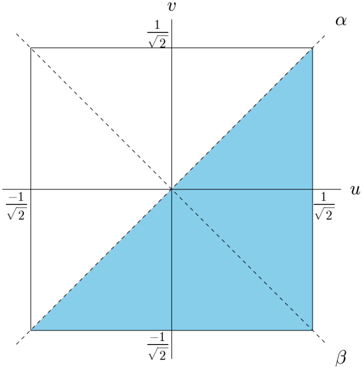

- Rotated square $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \times \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$:

$$\begin{cases} u = \dfrac{\beta + \alpha}{\sqrt{2}} & , \\ v = \dfrac{\alpha - \beta}{\sqrt{2}} & . \end{cases} \tag{15}$$

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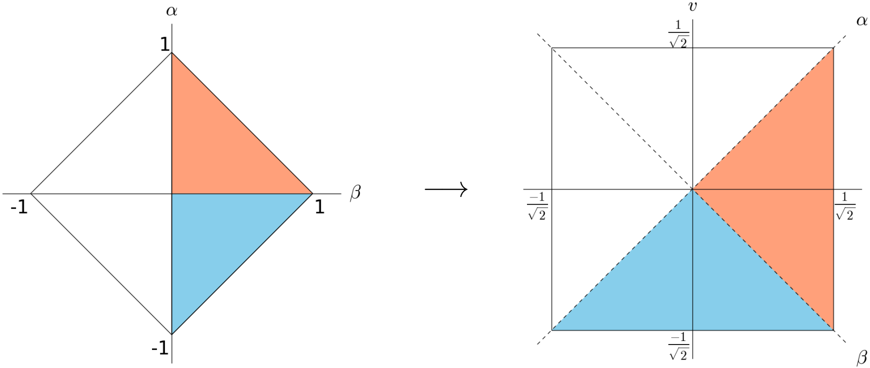

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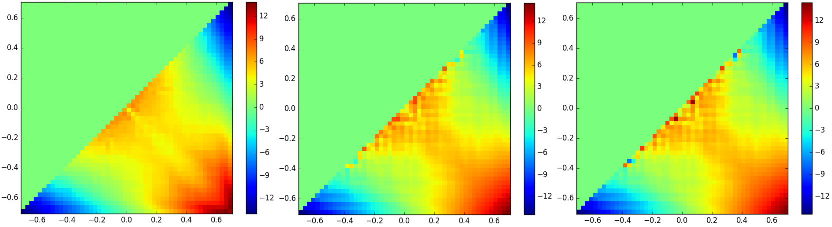
- α-parity of the DD:

$$f(\beta, -\alpha) = f(\beta, \alpha) \ . \tag{16}$$

First result



- Real application to a DSE toy model.

$$f(\beta,\alpha) = \begin{cases} ? & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

$$\downarrow$$

$$H(x,\xi)|_{x>|\xi|} = 30\,\frac{(1-x)^2\,(x^2-\xi^2)}{(1-\xi^2)^2}$$

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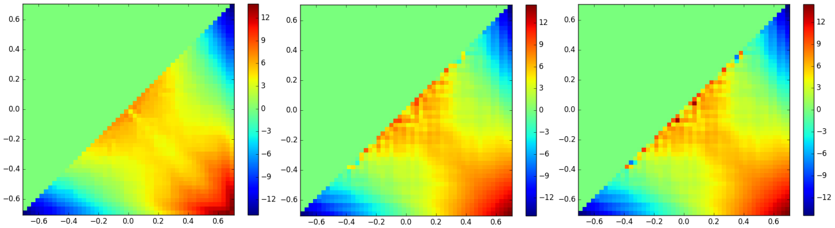

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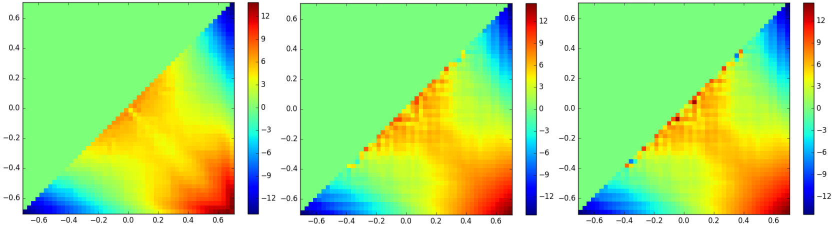

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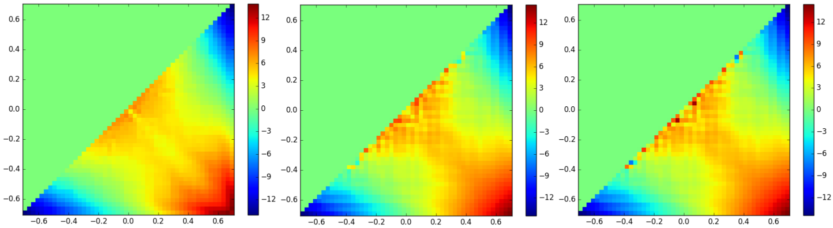

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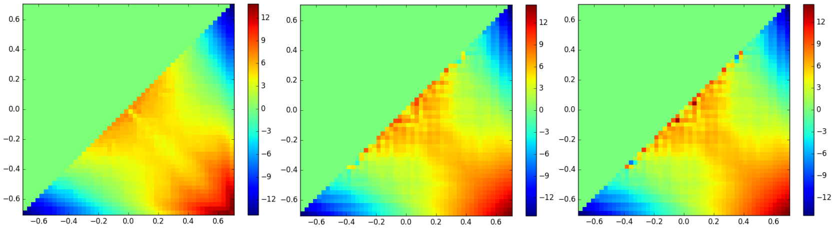

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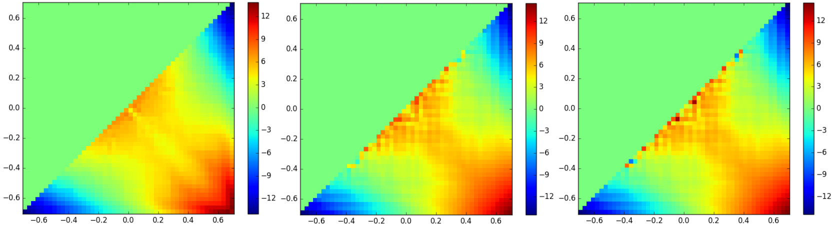

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$$f(\beta,\alpha) = \begin{cases} \frac{15}{2}\left(1 - 3\alpha^2 - 2\beta + 3\beta^2\right) & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

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- Least-squares problem:
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- Smooth function in Pobylitsa gauge:

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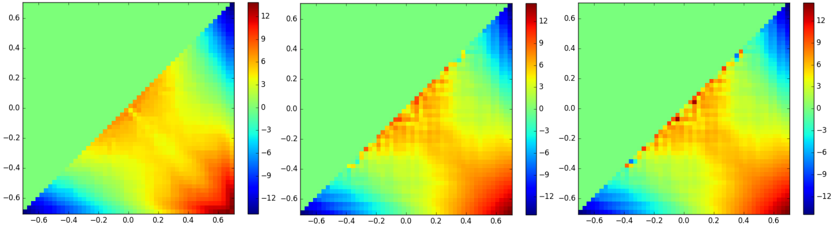

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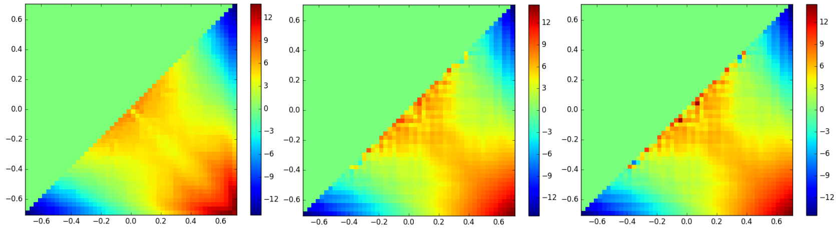

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  - Gauge introduced for positivity.

$$f(\beta,\alpha) = \begin{cases} \frac{15}{2}\left(1 - 3\alpha^2 - 2\beta + 3\beta^2\right) & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

$$\downarrow$$

$$H(x,\xi)\big|_{x>|\xi|} = 30\,\frac{(1-x)^2\left(x^2-\xi^2\right)}{\left(1-\xi^2\right)^2}$$

Quantitative comparison of DDs (DSE model)

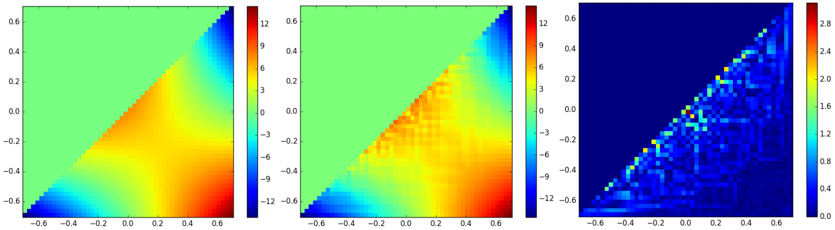

Figure: Quantitative comparison for the DSE toy model DD. Left: Theoretical discretized DD. Middle: Numerical solution at tolerance 10^{-6}. Right: Absolute difference.

$$f(\beta,\alpha) = \begin{cases} \frac{30}{4}\left(1 - 3\alpha^2 - 2\beta + 3\beta^2\right) & \beta > 0 \\ 0 & \beta < 0 \end{cases}$$

Quantitative comparison of GPDs (DSE model)

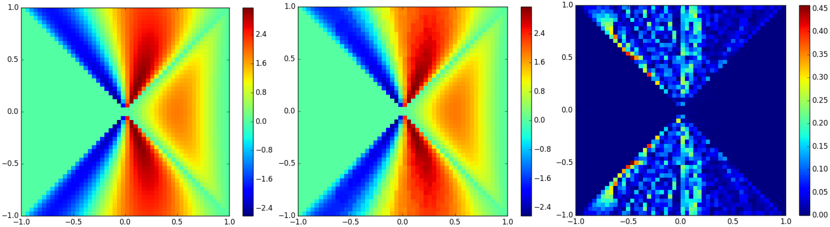

Figure: Quantitative comparison for the DSE toy model GPD obtained from the numerical DD solution. Left: Theoretical GPD. Middle: Numerical solution at tolerance 10^{-6}. Right: Absolute difference.

$$H(x,\xi) = \begin{cases} 30\dfrac{(1-x)^2\left(x^2-\xi^2\right)}{\left(1-\xi^2\right)^2} & x > |\xi| \\[2ex] \dfrac{15\,(x-1)\left(x^2-\xi^2\right)\left(\xi^2+2\,|\xi|\,x+x\right)}{2\,|\xi|^3\,(|\xi|+1)^2} & |x| < |\xi| \end{cases}$$

DDs & GPDs (DSE model)

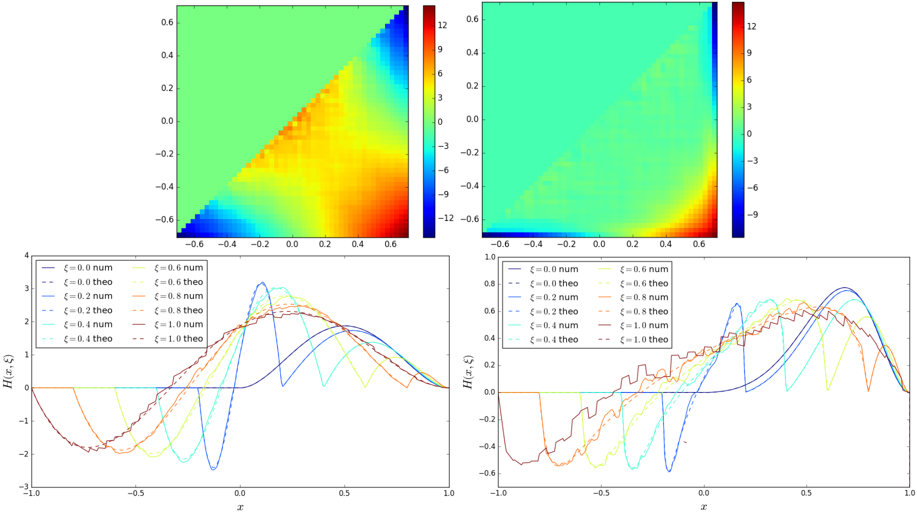

Figure: DDs and GPDs for the DSE model. Left: $t = 0$ GeV^2. Right: $t = 1$ GeV^2.

DDs & GPDs (Gaussian model)

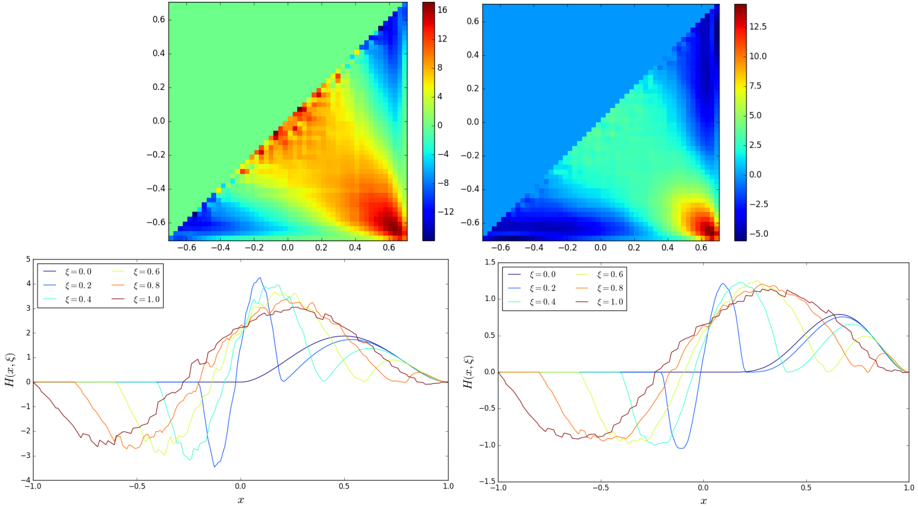

Figure: DDs and GPDs for a Gaussian toy model (similar to AdS/QCD). Left: $t = 0$ GeV^2. Right: $t = 1$ GeV^2.

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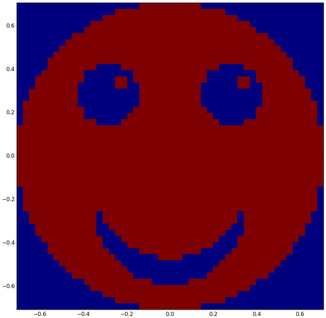
- Thank you!



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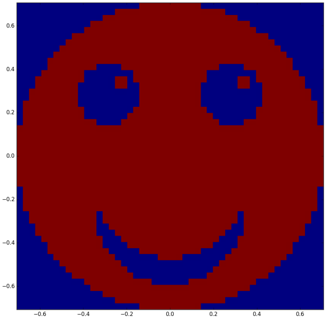
  - Any questions?



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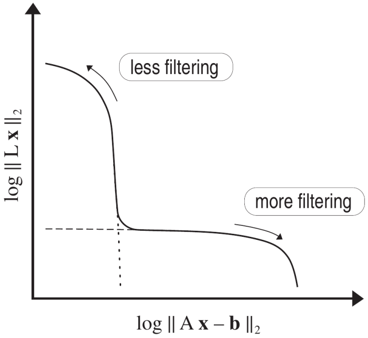
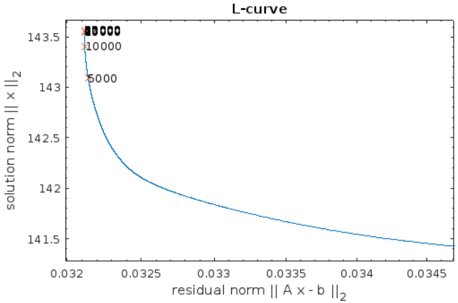
Discrete ill-posed problem



Theoretical “L-curve”: curve parameterized by the regularization factor.

(fig. taken from Ref. (Hansen, 2007))



L-curve with the iteration number as regularization factor.