# On the Unruh effect

(from a textbook)

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October 8th 2019

# Unruh effect (W.G. Unruh, Notes on black-hole evaporation, PRD 14 (1976) 870 sect III)

**But start with textbook treatments heavily relying on :**

**1) V. F. Mukhanov and S. Winitzki : Introduction to Quantum Fields in Classical Backgrounds (Chapter 8 The Unruh effect)**

and a tiny bit on :

2) R. M. Wald : Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics (Chapter 5 The Unruh effect) Chicago Lectures in Physics

3) R. M. Wald : The History and Present Status of Quantum Field Theory in Curved Spacetime gr-qc/0608018

4) K. Thorne, R.H. Price, D.A.MacDonald : Black holes the membrane paradigm, Yale University Press (Chap VIII.B6)

Thanks to Jean François Glicenstein for pointing 1) and 4) to me

# Unruh effect summarized

- in quantum field theory particles are excitations of quantum fields

- Unruh effect predicts that particles will be detected in a vacuum by an accelerated observer

- simplest case : observer moves with constant acceleration through Minkowski spacetime and measures the number of particles in a massless scalar field

- even though the field is in the vacuum state → observer finds a distribution of particles characteristic of a thermal bath of blackbody radiation

# Unruh effect

consider the trajectory of an observer moving with constant acceleration in Minkowski spacetime

acceleration of observer in its own frame of reference (proper acceleration) is constant

introduce several reference frames :

**- laboratory frame**
the usual inertial reference frame with the coordinates (t, x, y, z)

**- proper frame**
the accelerated system of reference that moves together with the observer
also called the **accelerated frame**

**- comoving frame :** defined at a time $t_0$
is the inertial frame in which the accelerated observer is instantaneously at rest at $t = t_0$
thus the term comoving frame actually refers to a different frame for each $t_0$

# Unruh effect : trajectory of accelerated observer

- consider a uniformly accelerated observer with a time-independent proper acceleration equal to a given 3-vector **a**

- trajectory of such an observer may be described by a worldline $x^\mu(\tau)$
  → $\tau$ proper time measured by the observer

- 4-acceleration in the laboratory frame (see appendix)

$$a^\mu \equiv \frac{d u^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} \qquad \text{with} \qquad u^\mu \equiv \frac{d x^\mu}{d\tau} \quad \text{and} \quad u^\mu u_\mu = 1$$

related to the three-dimensional proper acceleration **a** by

$$a^\mu a_\mu = -\,|\boldsymbol{a}|^2$$

# Unruh effect : trajectory of accelerated observer

- assume now that the acceleration is parallel to the x axis $\mathbf{a} \equiv (a, 0, 0)$ with $a > 0$ and that the observer moves only in the x direction → then (see appendix) :

$$x(\tau) = x_0 - \frac{1}{a} + \frac{1}{a} \cosh a\tau$$

$$t(\tau) = t_0 + \frac{1}{a} \sinh a\tau$$

- the trajectory has a simpler form if we choose the initial conditions $x(0) = a^{-1}$ and $t(0) = 0$

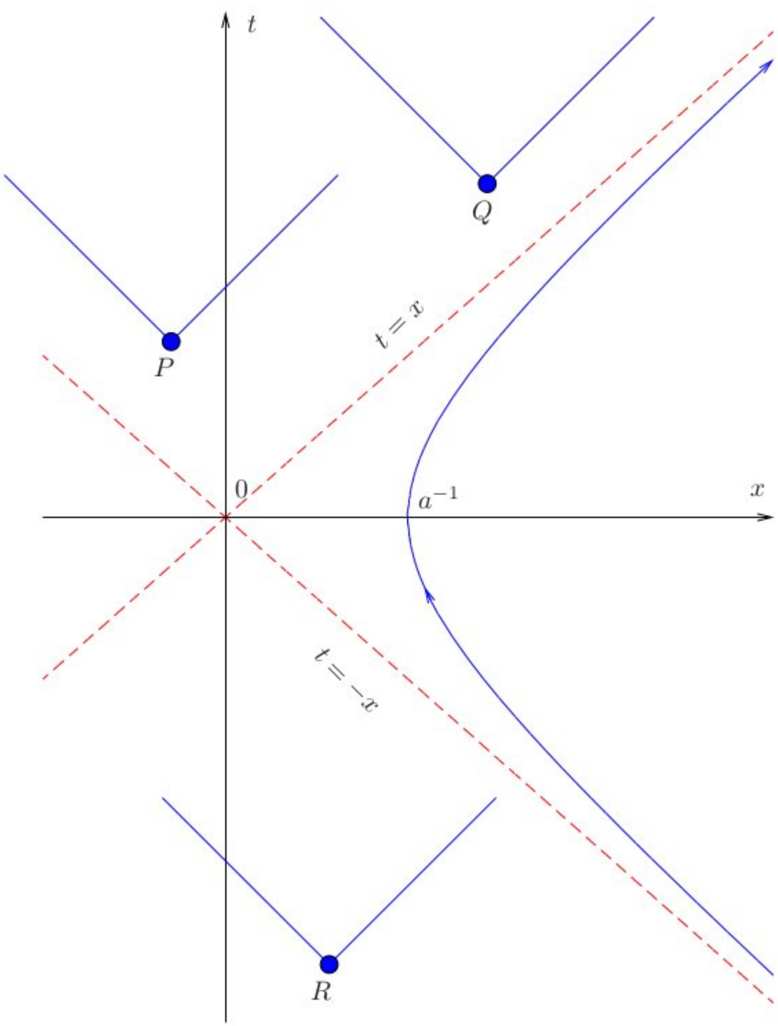
→ then the worldline is a branch of the hyperbola $\mathbf{x^2 - t^2 = a^{-2}}$

- this trajectory has zero velocity at $\tau = 0$ (which implies $x = x_0$, $t = t_0$)

# Unruh effect : trajectory of accelerated observer

- the worldline of the uniformly accelerated observer in the Minkowski spacetime is a branch of the hyperbola $x^2 - t^2 = a^{-2}$

- at large |t| the worldline approaches the lightcone




- the observer comes in from $x = +\infty$ decelerates and stops at $x = a^{-1}$ and then accelerates back towards infinity

- in the comoving frame of the observer, this motion takes infinite proper time, from $\tau = -\infty$ to $\tau = +\infty$.

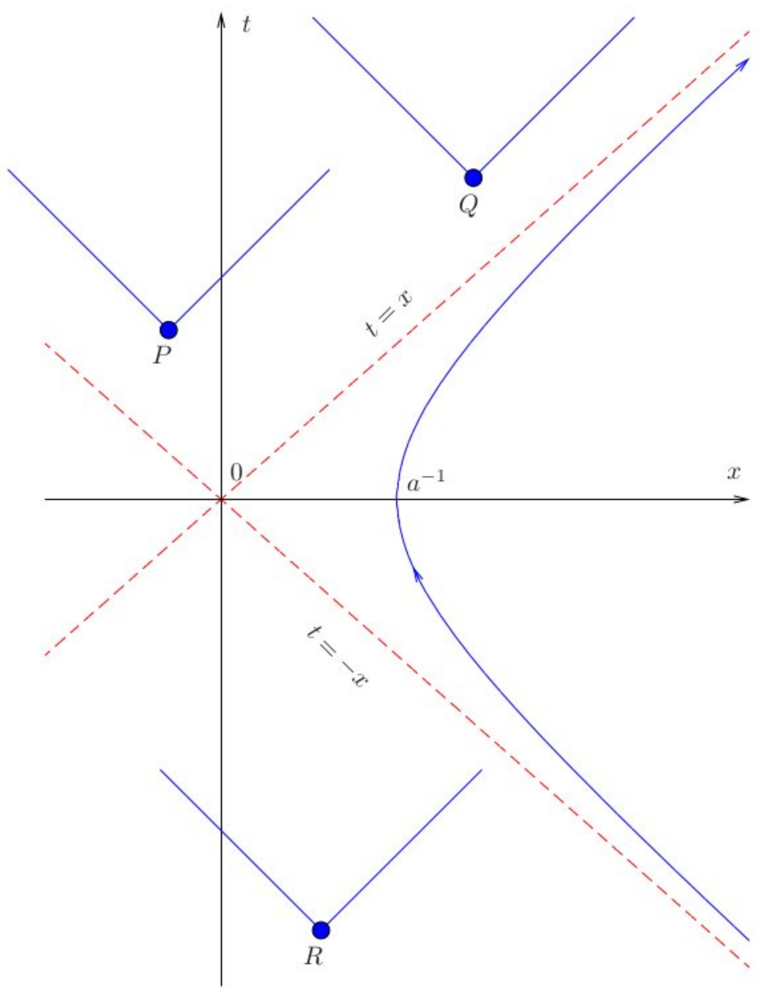
- dashed lines show the light-cone

- observer cannot receive any signals from the events P , Q and cannot send signals to R

## Unruh effect : horizon

- an accelerated observer cannot measure distances longer than $a^{-1}$ in the direction opposite to acceleration :

for instance the distances to the events P and Q




**one says that the accelerated observer perceives a horizon at proper distance $a^{-1}$**

# Unruh effect : proper coordinates

to describe quantum fields as seen by an accelerated observer

→ need to use the proper coordinates *($\tau$, $\xi$)*

where $\tau$ is the proper time and $\xi$ is the distance measured by the observer

the proper coordinate system ($\tau$, $\xi$) is related to the laboratory frame ($t$, $x$) by some transformation functions $\tau(t, x)$ and $\xi(t, x)$

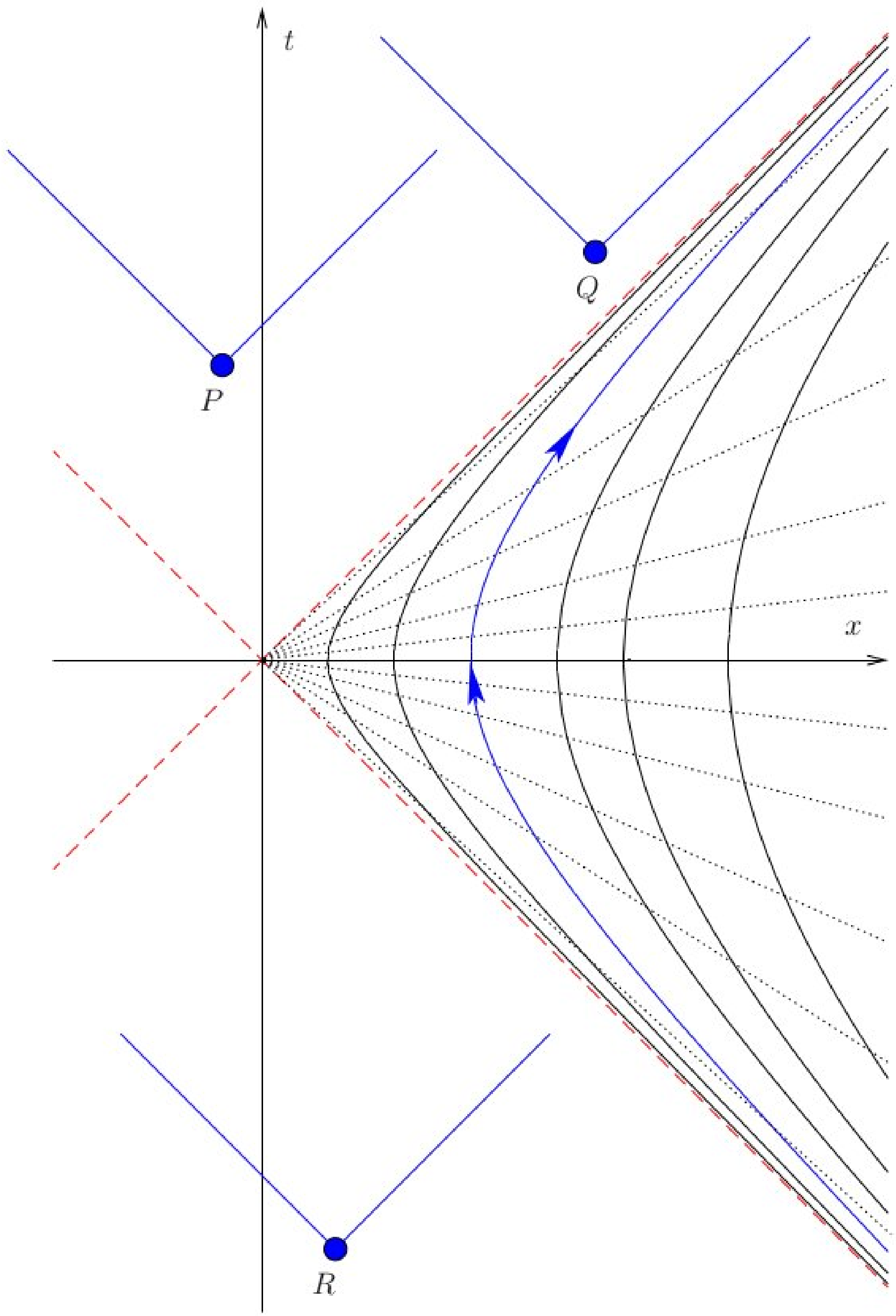
$$t(\tau, \xi) = \frac{1 + a\xi}{a} \sinh a\tau$$

$$x(\tau, \xi) = \frac{1 + a\xi}{a} \cosh a\tau$$

coordinates ($\tau$, $\xi$) vary in the intervals $-\infty < \tau < +\infty$ and $-a^{-1} < \xi < +\infty$

for $\xi < -a^{-1}$ → $\partial t/\partial\tau < 0$ i.e. the direction of time is opposite to that of $\tau$

# Unruh effect : horizon




proper coordinate system of a uniformly accelerated observer in the Minkowski spacetime

- solid hyperbolae are lines of constant proper distance $\xi$

- hyperbola with arrows is the worldline of observer $\xi = 0$ or $x^2 - t^2 = a^{-2}$

- lines of constant $\tau$ are dotted

- dashed lines show lightcone which corresponds to $\xi = -a^{-1}$

**events P , Q , R are not covered by the proper coordinate system**

**subdomain $x > |t|$ → Minkowski wedge**

# Unruh effect : Rindler spacetime

Minkowski metric in the proper coordinates $(\tau, \xi)$ is :

$$ds^2 = dt^2 - dx^2 = (1 + a\,\xi)^2\ d\tau^2 - d\xi^2$$

spacetime with this metric is called **Rindler spacetime**

curvature of Rindler spacetime is everywhere zero since it differs from Minkowski spacetime merely by a change of coordinates.

to develop quantum field theory in Rindler spacetime, we first rewrite the metric in a **conformally flat form** i.e. choosing a new spatial coordinate :

$$\tilde{\xi} \equiv \ln(1 + a\,\xi)$$

$\tilde{\xi}$ is called the conformal distance

so that we obtain a **common factor in the metric**

$$ds^2 = e^{2a\tilde{\xi}}\left(d\tau^2 - d\tilde{\xi}^2\right)$$

the proper distance $\xi$ is constrained by $\xi > -a^{-1}$ then the conformal distance varies in the interval $[-\infty, +\infty]$

relation between the laboratory coordinates and the conformal coordinates is

$$t(\tau, \tilde{\xi}) = a^{-1}\ e^{a\tilde{\xi}}\ \sinh a\,\tau \qquad\qquad x(\tau, \tilde{\xi}) = a^{-1}\ e^{a\tilde{\xi}}\ \cosh a\,\tau$$

# Unruh effect : quantization of a scalar field

- quantize a scalar field in the proper reference frame of a uniformly accelerated observer
  simplify problem → consider a massless scalar field in 1+1-dimensional spacetime

- **procedure of quantization is formally the same in both coordinate systems i.e. laboratory and accelerated frames :**

**- mode expansion in the laboratory frame**

annihilation operator　　　creation operator

$$\hat{\phi}(t,x) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[ e^{-i|k|t + ikx} \hat{a}_k^- + e^{i|k|t - ikx} \hat{a}_k^+ \right]$$

vacuum state in the laboratory frame i.e. the **Minkowski vacuum** **$|0_M\rangle$**
is the zero eigenvector of all the annihilation operators $\hat{a}^-_k$ **$\hat{a}^-_k |0_M\rangle = 0$** for all k

**- mode expansion in the accelerated frame**

$$\hat{\phi}(\tau,\tilde{\xi}) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^{1/2}} \frac{1}{\sqrt{2|k|}} \left[ e^{-i|k|\tau + ik\tilde{\xi}} \hat{b}_k^- + e^{i|k|\tau - ik\tilde{\xi}} \hat{b}_k^+ \right]$$

vacuum state in the accelerated frame i.e. the **Rindler vacuum** **$|0_R\rangle$**
is defined by **$\hat{b}^-_k |0_R\rangle = 0$** for all k

# Unruh effect : different vacuum

- mode expansions are decompositions into linear combinations of 2 different sets of basis functions with operators $\hat{a}^{\pm}_{k}$ and $\hat{b}^{\pm}_{k}$

- operators $\hat{a}_{k}$ and $\hat{b}_{k}$ are different although they satisfy similar commutation relations

→ **Rindler vacuum $|0_R>$ and Minkowski vacuum $|0_M>$ are 2 different quantum states of the field $\Phi$**

# Unruh effect : which is the correct vacuum ?

**which of the state $|0_M\rangle$ or $|0_R\rangle$ is the "correct" vacuum ?**

- observers accelerating would agree that the field in the state $|0_R\rangle$ has the lowest possible energy and the Minkowski state $|0_M\rangle$ has a higher energy

  → thus a particle detector at rest in the accelerated frame will register particles when the scalar field is in the state $|0_M\rangle$

- however in the laboratory frame the state with the lowest energy is $|0_M\rangle$ and the state $|0_R\rangle$ has a higher energy

  → therefore the Rindler vacuum state $|0_R\rangle$ will appear to be an excited state when examined by observers in the laboratory frame

**neither of the two vacuum states is "more correct" if considered by itself**

ultimately the choice of vacuum is determined by experiment:
the correct vacuum state must be such that the theoretical predictions agree with the available experimental data

# Light cone mode expansion : density of particles and Unruh temperature

- one can relate the two sets of operators $\hat{a}^{\pm}_{k}$ and $\hat{b}^{\pm}_{k}$

  → Bogolyubov transformations

  (after some gymnastic rewritting the previous mode expansions in lightcone coordinates - see appendix)

  → transformations linking the vacuum states of the quantum field in the Rindler frame and the Minkowksi frame (i.e. accelerated frame and laboratory frame)

- vacua $|0_M>$ and $|0_R>$ corresponding to operators $\hat{\mathbf{a}}^{-}$ and $\hat{\mathbf{b}}^{-}$ are different

  → the **a**-vacuum is a state with **b**-particles and vice versa

- one can compute the mean density of massless **b**-particle of energy E in the **a**-vacuum (from the Bogolyubov transformation coefficients - see appendix) :

$$n(E) = \left[ \exp\left( \frac{E}{T} \right) - 1 \right]^{-1} \qquad \text{with} \qquad T \equiv \frac{a}{2\pi}$$

acceleration

thermal spectrum

Unruh temperature

## Unruh effect : one physical interpretation

a physical interpretation of the Unruh effect as seen in the laboratory frame is the following :

- the accelerated detector is coupled to the quantum fields and perturbs their quantum state around its trajectory

- this perturbation is very small but as a result the detector registers particles although the fields were previously in the vacuum state

- the detected particles are real and the energy for these particles comes from the agent that accelerates the detector

# finishing with an exercise

## Unruh effect : exercice

a glass of water is moving with constant acceleration

what is the smallest acceleration that would make the water boil due to the Unruh effect ?

# Unruh effect : exercice

a glass of water is moving with constant acceleration

what is the smallest acceleration that would make the water boil due to the Unruh effect ?

expressing all quantities in SI units :

$$T \equiv \frac{a}{2\pi} \qquad \text{becomes} \qquad k\,T \equiv \frac{\hbar}{c}\,\frac{a}{2\pi}$$

where k ≈ 1.38 $10^{-23}$ J/K is the Boltzmann's constant

the boiling point of water is T = 373 K

**so the required acceleration is a ≈ $10^{22}$ m/s$^2$**

**the Unruh effect is quite difficult to use in practice because the acceleration required to produce a measurable temperature is enormous**

# Possible next steps

- Hawking like radiation from accelerated mirrors (from an analog/similar framework)

- Hawking radiation from Schwarzschild BH

- ‘t Hooft approach : scattering matrix approach, black hole unitarity, back reaction,
  antipodal entanglement

- what about a possible role of BMS (Bondi, van der Burg, Metzner, Sachs)
  asymptotic symmetries, relation to soft hairs on BH
  Strominger at al. (+ one of the last papers from S.W. Hawking with collaborators) ?

- …………………………………..

- how to count BH micro-states ?

- ………………………………….

# APPENDIX

**Derivation of Eq. (8.2).** Let $u^\mu(\tau)$ be the observer's 4-velocity and let $t_c$ be the time variable in the comoving frame defined at $\tau = \tau_0$; this is the time measured by an *inertial* observer moving with the constant velocity $u^\mu(\tau_0)$. We shall show that the 4-acceleration $a^\mu(\tau)$ in the comoving frame has components $\left(0, a^1, a^2, a^3\right)$, where $a^i$ are the components of the acceleration 3-vector $\mathbf{a} \equiv d^2\mathbf{x}/dt_c^2$ measured in the comoving frame. It will then follow that Eq. (8.2) holds in the comoving frame, and hence it holds also in the laboratory frame since the Lorentz-invariant quantity $a^\mu a_\mu$ is the same in all frames.

Since the comoving frame moves with the velocity $u^\mu(\tau_0)$, the 4-vector $u^\mu(\tau_0)$ has the components $(1, 0, 0, 0)$ in that frame. The derivative of the identity $u^\mu(\tau)u_\mu(\tau) = 1$ with respect to $\tau$ yields $a^\mu(\tau)u_\mu(\tau) = 0$, therefore $a^0(\tau_0) = 0$ in the comoving frame. Since $dt_c = u^0(\tau)d\tau$ and $u^0(\tau_0) = 1$, we have

$$\frac{d^2x^\mu}{dt_c^2} = \frac{1}{u^0}\frac{d}{d\tau}\left[\frac{1}{u^0}\frac{dx^\mu}{d\tau}\right] = \frac{d^2x^\mu}{d\tau^2} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\frac{1}{u^0}.$$

It remains to compute

$$\frac{d}{d\tau}\frac{1}{u^0(\tau_0)} = -\left[u^0(\tau_0)\right]^{-2}\left.\frac{du^0}{d\tau}\right|_{\tau=\tau_0} = -a^0\left(\tau_0\right) = 0,$$

and it follows that $d^2x^\mu/d\tau^2 = d^2x^\mu/dt_c^2 = \left(0, a^1, a^2, a^3\right)$ as required. (Self-test question: why is $a^\mu = du^\mu/d\tau \neq 0$ even though $u^\mu = (1, 0, 0, 0)$ in the comoving frame?)

**Derivation of Eq. (8.3).** Since $a^\mu = du^\mu/d\tau$ and $u^2 = u^3 = 0$, the components $u^0$, $u^1$ of the velocity satisfy

$$\left(\frac{du^0}{d\tau}\right)^2 - \left(\frac{du^1}{d\tau}\right)^2 = -a^2,$$
$$\left(u^0\right)^2 - \left(u^1\right)^2 = 1.$$

We may assume that $u_0 > 0$ (the time $\tau$ grows together with $t$) and that $du^1/d\tau > 0$, since the acceleration is in the positive $x$ direction. Then

$$u^0 = \sqrt{1 + (u^1)^2}; \quad \frac{du^1}{d\tau} = a\sqrt{1 + (u^1)^2}.$$

The solution with the initial condition $u^1(0) = 0$ is

$$u^1(\tau) \equiv \frac{dx}{d\tau} = \sinh a\tau, \quad u^0(\tau) \equiv \frac{dt}{d\tau} = \cosh a\tau.$$

After an integration we obtain Eq. (8.3).

## Unruh effect : horizon

- one can verify that an accelerated observer cannot measure distances longer than $a^{-1}$ in the direction opposite to acceleration :

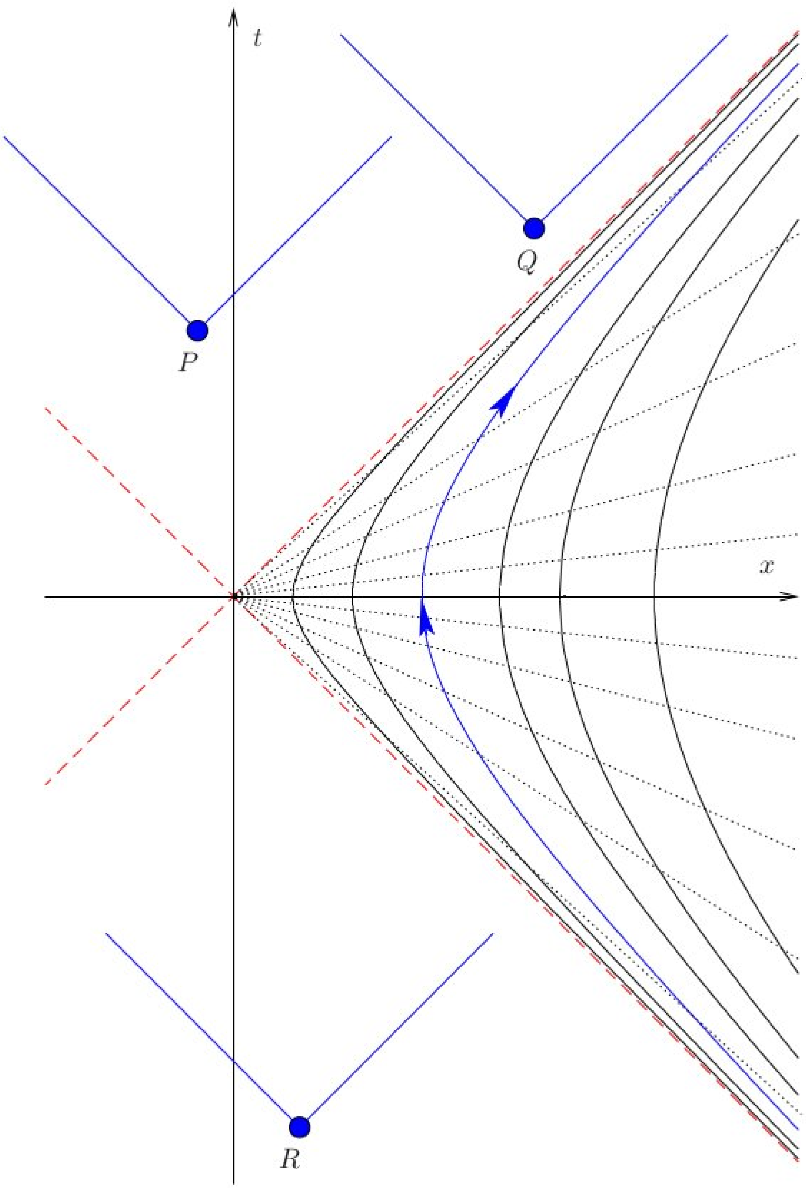
  for instance the distances to the events P and Q

- a measurement of the distance to a point requires to place a clock at that point and to synchronize that clock with the observer’s clock

- however the observer cannot synchronize clocks with the events P and Q because no signals can be ever received from these events

**one says that the accelerated observer perceives a horizon at proper distance $a^{-1}$**

# Unruh effect : horizon




- another way to see that the line $\xi = -a^{-1}$ is a horizon is to consider a line of constant proper length $\xi = \xi_0 > -a^{-1}$

line $\xi = \xi_0$ is a trajectory of the form $x^2 - t^2 = \text{const}$ with proper acceleration

$$a_0 \equiv \frac{1}{\sqrt{X^2 - t^2}} = \left(\xi_0 + a^{-1}\right)^{-1}$$

therefore worldline $\xi = -a^{-1}$ would have to represent an infinite proper acceleration which would require an infinitely large force and is thus impossible

it follows that an accelerated observer cannot hold a rigid measuring stick longer than $a^{-1}$ in the direction opposite to acceleration

(a rigid stick is one that would keep its proper distance constant in the observes's reference frame)

# Unruh effect : light cone mode expansion

- convenient to introduce the lightcone coordinates

laboratory frame : $\bar{u} \equiv t - x \quad , \quad \bar{v} \equiv t + x$

accelerated frame : $u \equiv \tau - \xi \quad , \quad v \equiv \tau + \xi$

- metric, field equations and their general solutions expressed more concisely in the lightcone coordinates

$$ds^2 = d\bar{u}\, d\bar{v} = e^{a(v-u)} du\, dv$$

$$\frac{\partial^2}{\partial \bar{u}\, \partial \bar{v}} \phi(\bar{u}, \bar{v}) = 0 \quad , \quad \phi(\bar{u}, \bar{v}) = A(\bar{u}) + B(\bar{v})$$

$$\frac{\partial^2}{\partial u\, \partial v} \phi(u, v) = 0 \quad , \quad \phi(u, v) = P(u) + Q(v)$$

# Light cone mode expansion : Minkowski frame (laboratory frame)

mode expansion can be rewritten in the coordinates $\bar{u}$, $\bar{v}$ by first splitting the integration into the ranges of positive and negative k

then introduce $\omega = |k|$ as integration variable with range $0 < \omega < +\infty$

lightcone mode expansions explicitly decompose the field $\hat{\phi}(\bar{u}, \bar{v})$ into a sum of functions of $\bar{u}$ and functions of $\bar{v}$ :

$$\hat{\phi}(\bar{u}, \bar{v}) = \hat{A}(\bar{u}) + \hat{B}(\bar{v})$$

$$\hat{A}(\bar{u}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[ e^{-i\omega\bar{u}} \hat{a}_{\omega}^{-} + e^{i\omega\bar{u}} \hat{a}_{\omega}^{+} \right]$$

$$\hat{B}(\bar{v}) = \int_0^{+\infty} \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[ e^{-i\omega\bar{v}} \hat{a}_{-\omega}^{-} + e^{i\omega\bar{v}} \hat{a}_{-\omega}^{+} \right]$$

# Light cone mode expansion : Rindler frame (accelerated frame)

lightcone mode expansion in Rindler frame has exactly the same form except for involving coordinates (u, v) instead of $(\bar{u}, \bar{v})$

use integration variable Ω to distinguish Rindler frame expansion from that of Minkowski frame

$$\hat{\phi}(u, v) = \hat{P}(u) + \hat{Q}(v)$$

$$\hat{P}(u) = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega u} \hat{b}^-_{\Omega} + e^{i\Omega u} \hat{b}^+_{\Omega} \right]$$

$$\hat{Q}(v) = \int_0^{+\infty} \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[ e^{-i\Omega v} \hat{b}^-_{-\Omega} + e^{i\Omega v} \hat{b}^+_{-\Omega} \right]$$

## Light cone mode expansion : Rindler/Minkowski frames relation

relations between laboratory (Minkowski) frame ($\bar{u}$, $\bar{v}$ coordinates) and accelerated (Rindler) frame (u, v coordinates) are simpler

$$\bar{u} = -a^{-1}\ e^{-au} \qquad \bar{v} = a^{-1}\ e^{-av}$$

this coordinate transformation does not mix u and v so that

$$\hat{\phi}(u,v) = \hat{A}\left(\bar{u}(u)\right) + \hat{B}\left(\bar{v}(v)\right) = \hat{P}(u) + \hat{Q}(v)$$

entails two separate relations for u and for v

$$\hat{A}\left(\bar{u}(u)\right) = \hat{P}(u) \qquad \hat{B}\left(\bar{v}(v)\right) = \hat{Q}(v)$$

# Unruh effect : Bogolyubov transformations (I)

relations between operators $\hat{a}^{\pm}_{\pm\omega}$ and $\hat{b}^{\pm}_{\pm\Omega}$ are Bogolyubov transformations

they are obtained from these two separate relations for u and for v

$$\hat{A}(\bar{u}) = \hat{P}(u) \qquad \hat{B}(\bar{v}) = \hat{Q}(v)$$

operators $\hat{a}^{\pm}_{\omega}$ with positive momenta $\omega$ are expressed through $\hat{b}^{\pm}_{\Omega}$ with positive momenta $\Omega$

while operators $\hat{a}^{\pm}_{-\omega}$ are expressed through negative-momentum operators $\hat{b}^{\pm}_{-\Omega}$

there is no mixing between operators of positive and negative momentum

# Unruh effect : Bogolyubov transformations (II)

for example from : $\hat{A}(\bar{u}) = \hat{P}(u)$

$$\hat{b}^{-}_{\Omega} = \int_0^{+\infty} d\,\omega \left[ \alpha_{\omega\,\Omega}\ \hat{a}^{-}_{\omega} + \beta_{\omega\,\Omega}\ \hat{a}^{+}_{\omega} \right]$$

with Bogolyubov coefficients

$$\alpha_{\omega\,\Omega} = \sqrt{\frac{\Omega}{\omega}}\ F(\omega,\ \Omega)$$

$$\beta_{\omega\,\Omega} = \sqrt{\frac{\Omega}{\omega}}\ F(-\omega,\ \Omega)$$

$$\omega > 0\ ,\quad \Omega > 0$$

$\hat{b}^{+}_{\Omega}$ expressed through $\hat{a}^{\pm}_{\omega}$ using hermitian conjugate of $\hat{b}^{-}_{\Omega}$ above and using $F^{*}(\omega,\ \Omega) = F(-\omega,\ -\Omega)$

with auxiliary function : $F(\omega,\Omega) = \int_{-\infty}^{+\infty} \frac{du}{2\,\pi}\ e^{i\Omega\ u\ -\ i\,\omega\ \bar{u}} = \int_{-\infty}^{+\infty} \frac{du}{2\,\pi}\ \exp\left[\ i\,\Omega\ u\ -\ i\,\frac{\omega}{a}\ e^{-au}\ \right]$

Bogolyubov transformations mixing creation and annihilation operators with different momenta $\omega \neq \Omega$

# Unruh effect : Bogolyubov transformations (III)

analogously relations between operators $\hat{a}^{\pm}_{-\omega}$ and $\hat{b}^{\pm}_{-\Omega}$ are obtained from

$$\hat{B}\left(\bar{v}\right) = \hat{Q}\left(v\right)$$

i.e. the results for negative momenta are completely analogous

# Unruh effect : density of particles (I)

vacua $|0_M\rangle$ and $|0_R\rangle$ corresponding to operators $\hat{a}^-_\omega$ and $\hat{b}^-_\Omega$ are different
→ the a-vacuum is a state with b-particles and vice versa

what is the density of b-particles in the a-vacuum state ?

b-particle number operator is : $\hat{N}_\Omega \equiv \hat{b}^+_\Omega \hat{b}^-_\Omega$
→ the average b-particle number in the a-vacuum $|0_M\rangle$ is equal to the expectation value of $\hat{N}_\Omega$ :

$$\langle \hat{N}_\Omega \rangle \equiv \langle 0_M | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_M \rangle$$

$$= \langle 0_M | \int d\omega \left[ \alpha^*_{\omega\,\Omega} \hat{a}^+_\omega + \beta^*_{\omega\,\Omega} \hat{a}^-_\omega \right] \int d\omega' \left[ \alpha_{\omega'\,\Omega} \hat{a}^-_{\omega'} + \beta_{\omega'\,\Omega} \hat{a}^+_{\omega'} \right] | 0_M \rangle$$

$$= \int d\omega \, |\beta_{\omega\,\Omega}|^2$$

# Unruh effect : density of particles (II)

computing the integral yield:

$$<\hat{N}_\Omega> = \left[ \exp\left( \frac{2\pi \, \Omega}{a} \right) - 1 \right]^{-1} \delta(0) \equiv n_\Omega \, \delta(0)$$

where

$$n_\Omega = \left[ \exp\left( \frac{2\pi \, \Omega}{a} \right) - 1 \right]^{-1}$$

is the mean density of particle with momentum $\Omega$

## Unruh effect : Unruh temperature

a massless particle with momentum $\Omega$ has energy $E = |\,\Omega\,|$

so the mean density of particle $n_\Omega$ :

$$n_\Omega = \left[ \exp\left( \frac{2\pi\,\Omega}{a} \right) - 1 \right]^{-1}$$

is equivalent to the Bose-Einstein distribution :

$$n(E) = \left[ \exp\left( \frac{E}{T} \right) - 1 \right]^{-1}$$

where T is the Unruh temperature :

$$T \equiv \frac{a}{2\pi}$$

# BACKUP

**Abstract**

Quantum field theory in curved spacetime is a theory wherein matter is treated fully in accord with the principles of quantum field theory, but gravity is treated classically in accord with general relativity. It is not expected to be an exact theory of nature, but it should provide a good approximate description in circumstances where the quantum effects of gravity itself do not play a dominant role. Some of the earliest applications of the theory were to study particle creation effects in an expanding universe. A major impetus to the theory was provided by Hawking's calculation of particle creation by black holes, showing that black holes radiate as perfect black bodies. During the past 30 years, considerable progress has been made in giving a mathematically rigorous formulation of quantum field theory in curved spacetime. Major issues of principle with regard to the formulation of the theory arise from the lack of Poincare symmetry, the absence of a preferred vacuum state, and, in general, the absence of asymptotic regions in which particle states can be defined. By the mid-1980's, it was understood how all of these difficulties could be overcome for free (i.e., non-self-interacting) quantum fields by formulating the theory via the algebraic approach and focusing attention on the local field observables rather than a notion of "particles". However, these ideas, by themselves, were not adequate for the formulation of interacting quantum field theory, even at a perturbative level, since standard renormalization prescriptions in Minkowski spacetime rely heavily on Poincare invariance and the existence of a Poincare invariant vacuum state. However, during the past decade, great progress has been made, mainly due to the importation into the theory of the methods of "microlocal analysis". This article will describe the historical development of the subject and describe some of the recent progress.

The particle interpretation/description of quantum field theory in flat spacetime has been remarkably successful—to the extent that one might easily get the impression from the way the theory is normally described that, at a fundamental level, quantum field theory is really a theory of particles. However, the definition of particles relies on the decomposition of $\phi$ into annihilation and creation operators in eq.(12). This decomposition, in turn, relies heavily on the time translation symmetry of Minkowski spacetime, since the "annihilation part" of $\phi$ is its positive frequency part with respect to time translations. In a curved spacetime that does not possess a time translation symmetry, it is far from obvious how a notion of "particles" should be defined.

The Unruh effect may appear paradoxical to readers who are used to thinking that quantum field theory is, fundamentally, a theory of "particles", and that the notion of "particles" has objective significance. How can an accelerating observer assert that "particles" are present in region I when any inertial observer would assert that, "in reality", all of Minkowski spacetime is devoid of particles? Which of these two observers is "correct" in his assertion? The answer, of course, is that both observers are correct: It simply happens that the natural notion of "particles" defined by accelerating observers (convenient for characterizing the behavior of "particle detectors"—like the model of section 3.3—which are "time translationally invariant" with respect to $b^a$) differs from the natural notion of particles defined by inertial observers (convenient for characterizing the behavior of detectors which are invariant under ordinary time translations). No paradox arises when one views quantum field theory as, fundamentally, being a theory of local field observables, with the notion of "particles" merely being introduced as a convenient way of labeling states in certain situations.

R. Wald : The Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics (Chapter 5 The Unruh effect) Chicago Lectures in Physics

Nevertheless, the same physical predictions must be obtained whether one labels the states of the quantum field theory via the natural labeling of the accelerating observer or that of the inertial observer. Now, when the field is in the Minkowski vacuum state, $|0>_1$,—and, thus, is in the thermal density matrix (5.1.28) in the natural labeling given by the accelerating observer—there is a nonzero probability that a particle detector carried by the accelerating observer will make a transition to an excited state. The accelerating observer, of course, would describe this process as being simply the result of the absorption of a "particle" by his detector. The inertial observer must also see this transition in the state of the accelerating detector (as well as the accompanying change in the state of the field), and it is instructive to analyze how he would explain what has occurred. This can be worked out explicitly for the simple model particle detector considered in section 3.3. The result is that the inertial observer would describe this process as the emission of a "particle" by the "detector", accompanied by a change in the state of the detector due to "radiation reaction". Further details of the inertial description of this process can be found in Unruh and Wald (1984).

R. Wald : The Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics (Chapter 5 The Unruh effect) Chicago Lectures in Physics

# K. Thorne, R.H. Price, D.A.MacDonald
## Black holes the membrane paradigm (chap VIII)

- Unruh discovered that an accelerated particle detector in flat, empty spacetime should behave as though it were bathed in a perfect bath of thermal radiation with temperature $T = (h/4\pi^2)\, a/ k_B$ where a is the detector's acceleration.

- Since a static observer (FIDO) just above Schwarzschild horizon can be viewed, in the Rindler approximation, as completely analogous to an accelerated observer in flat spacetime with acceleration $a = g_H \alpha$ , Unruh's insight suggested that such a FIDO should feel himself bathed in thermal radiation with locally measured temperature $T = (h/4\pi^2)\, (g_H/\alpha) / kB = T_H /\alpha$

- This thermal radiation (« thermal atmosphere of the hole »), climbing up through the hole’s gravitational field, would be redshifted by a factor $\alpha$ and therefore would emerge with a temperature $T_H$ as Hawking's thermal emission.

K. Thorne, R.H. Price, D.A.MacDonald, Black holes the membrane paradigm., Yale University Press

K. Thorne, R.H. Price, D.A.MacDonald
Black holes the membrane paradigm (chap VIII)

- not easy to bring physical intuition into accord with these quantum field theory predictions

- especially troubling was the fact that, an accelerated observer in flat spacetime sees a thermal bath, freely falling observers see pure vacuum

- correspondingly, although static FIDOs near a Schwarzschild black hole see a thermal atmosphere, freely falling observers see no such atmosphere at all

- considerable progress toward understanding these apparently contradictory viewpoints came from a series of model problems invented and studied by Unruh and Wald (1982, 1984)

K. Thorne, R.H. Price, D.A.MacDonald, Black holes
the membrane paradigm., Yale University Press

# K. Thorne, R.H. Price, D.A.MacDonald
## Black holes the membrane paradigm (chap VIII)

- in one such problem they showed that when an accelerated observer absorbs a quantum from the surrounding thermal bath a freely falling observer sees him emit a quantum

- these contrasting perception were reconciled by showing that both observers agree that the absorption/emission has increased the energy in the radiation field

this is obvious from the freely falling observer’s viewpoint :
the field was empty before the emission and contains one quantum afterwards

it is less obvious but true from the accelerated observer’s viewpoint :

the field was in a perfectly thermal, mixed state – and had a finite probability for containing no quantum whatsoever – before the absorption

by absorbing a quantum, the accelerated observer performs a partial measurement on the field ; for example, he learns that it contains at least one quantum before the absorption

this partial measurement, despite the absorption, turns out to increase the expectation value of the energy in the field, as computed by the accelerated observer

K. Thorne, R.H. Price, D.A.MacDonald, Black holes
the membrane paradigm., Yale University Press

# K. Thorne, R.H. Price, D.A.MacDonald
Black holes the membrane paradigm (chap VIII)

A particle detector carried by an inertial observer in flat, empty spacetime detects no particles whatsoever ... correspondingly, the expectation value of the stress-energy tensor ($\langle T^{\mu\nu}\rangle$) … is precisely zero. It is this $\langle T^{\mu\nu}\rangle$ which presumably couples to gravity through the Einstein field equations and which, because it vanishes, leaves spacetime perfectly flat.

A uniformly accelerated observer in flat, empty spacetime moves along a hyperbola in the Minkowski spacetime and a family of such accelerated observers moves along a family of such hyperbolae

Because such a family cannot sample the entire spacetime (the FIDOS are confined to the right-hand quadrant in the Minkowski spacetime diagram) the family cannot make a sufficiently global measurement of any field so as to verify that it indeed is in the "Minkowski vacuum state" (is unexcited)'

As a result, such FIDOs can obtain only partial information about the state of the field - partial information which corresponds to regarding each mode of the field as in a mixed state, with a nonunit probability $P_o$ to have zero quanta, and non zero probability $P_1$, $P_2$, … to have one, two, …. quanta

…. these probabilities are precisely thermally distributed and moreover these thermal $P_n$ are precisely the probabilities that n quanta will be detected in the mode by a real physical particle detector

K. Thorne, R.H. Price, D.A.MacDonald, Black holes the membrane paradigm., Yale University Press

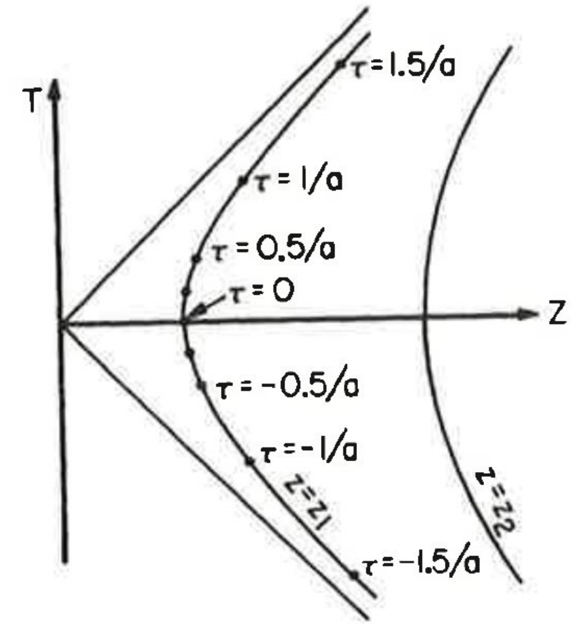
# K. Thorne, R.H. Price, D.A.MacDonald
Black holes the membrane paradigm (chap VIII)

- an accelerated detector behaves very differently from an unaccelerated detector and the energy eigenstates to which it couples with time-independent strength is very different from eigenstates of constant inertially measured energy

- in order to measure a quantum of energy $E^K \sim k_B T = h\, a/4\pi^2$ , which has, as seen by the accelerated detector, a frequency $\sigma^K = (2\pi/h)\, E^K \sim a/2\pi$ , the detector must make a continuous measurement that lasts longer than $\Delta r \sim \pi/\sigma^K \sim 2\pi^2/a \sim 20/a$

- as one sees from the figure, relative to an inertial observer the detector changes its own velocity by nearly the speed of light during this measurement time.

- with such radically changing velocity, the detector surely will not couple in any simple manner to the modes that an inertial observer regards as simple !




K. Thorne, R.H. Price, D.A.MacDonald, Black holes the membrane paradigm., Yale University Press

# K. Thorne, R.H. Price, D.A.MacDonald
## Black holes the membrane paradigm (chap VIII)

- the thermal bath, which the accelerated observers genuinely feel, is perfectly compatible with a vanishing value of $<T^{\mu\nu}>$

- the compatibility arises from the effects of vacuum polarization :

From the viewpoint of the accelerated observers, vacuum polarization gives a contribution to $<T^{\mu\nu}>$ precisely equal and opposite to that of a perfect thermal bath with locally measured temperature $T = (h/2\pi)\, a/k_B$

This contribution of vacuum polarization to $<T^{\mu\nu}>$ is independent of the actual state of the fields

K. Thorne, R.H. Price, D.A.MacDonald, Black holes the membrane paradigm., Yale University Press

# Glimpses of
# Quantum Field Theory
# in Curved Spacetime

# quantum field theory in curved spacetime

- quantum field theory in curved spacetime is the theory of quantum fields propagating in a classical curved spacetime.

- the spacetime is described in this case in accord with general relativity by a manifold M on which is defined a Lorentz metric $g_{ab}$

- in the framework of quantum field theory in curved spacetime, back-reaction of the quantum fields on the spacetime geometry can be taken into account by imposing the semi-classical Einstein equation $G_{ab} = 8\pi \langle T_{ab} \rangle$

- issues associated with back-reaction not considered
  → in the following
  (M, $g_{ab}$) may be taken to be an arbitrary, fixed globally hyperbolic spacetime

R. Wald : The History and Present Status of Quantum Field Theory in Curved Spacetime, gr-qc/0608018

# reminder

- much of the quantum theory of a free field follows directly from the analysis of an ordinary quantum mechanical harmonic oscillator described by the Hamiltonian

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 \; q^2$$

- introducing the "lowering" (or "annihilation") operator

$$a \equiv \sqrt{\frac{\omega}{2}} q^2 + i \sqrt{\frac{1}{2\,\omega}} \; p$$

- we can rewrite H as

$$H = \omega \left( a^+ \; a + \frac{1}{2} I \right)$$

- where $a^+$ is referred to as the "raising" (or "creation") operator and we have the commutation relations

$$\left[ a^+ , a \right] = I \quad , \quad \left[ H , a \right] = -\omega a$$

# reminder

- in the Heisenberg representation the position operator $q_H$ is given by

$$q_H = \sqrt{\frac{1}{2\omega}} \left( e^{-i\omega t} a + e^{i\omega t} a^+ \right)$$

- annihilation operator a is seen to be the positive frequency part of the position operator

- ground state $|0>$ of the harmonic oscillator is determined by

$$a|0> = 0$$

- all other states of the harmonic oscillator obtained by successive applications of $a^+$ to $|0>$

# reminder

- consider, now, a free Klein-Gordon scalar field $\Phi$ in Minkowski spacetime
- classically $\Phi$ satisfies the wave equation

$$\partial^a \partial_b \phi - m^2 \phi = 0$$

- to avoid technical awkwardness $\rightarrow$ convenient to imagine that the scalar field resides in a cubic box of side L with periodic boundary conditions

$$\phi_{\vec{k}} = L^{-3/2} \int e^{-i\vec{k}\cdot\vec{x}} \, \phi(t, \vec{x}) \, d^3 x \qquad \vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$$

- in that case $\Phi(t, \vec{x})$ can be decomposed in terms of a Fourier series in $\vec{x}$

$$H = \sum_{\vec{k}} \frac{1}{2} \left( |\dot{\phi}_{\vec{k}}|^2 + \omega_{\vec{k}}^2 \, |\phi_{\vec{k}}|^2 \right)$$

$$\omega_{\vec{k}}^2 = |\vec{k}|^2 + m^2$$

# reminder

- quantum field theory associated to $\Phi$ can be obtained by quantizing each of these oscillators

- Heisenberg field operator $\Phi(t, \vec{x})$ then given by

$$\phi(t, \vec{x}) = L^{-3/2} \sum_{\vec{k}} \frac{1}{2\,\omega_{\vec{k}}} \left( e^{i\vec{k}\cdot\vec{x} - i\,\omega_{\vec{k}} t} a_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x} + i\,\omega_{\vec{k}} t} a^{+}_{\vec{k}} \right)$$

- however the sum on the r.h.s does not converge in any sense that would allow one to define the operator $\Phi$ at the point $(t, \vec{x})$

→ roughly speaking, the infinite number of arbitrarily high frequency oscillators fluctuate too much to allow $\Phi(t, \vec{x})$ to be defined

→ difficulty can be overcome by "smearing" $\Phi$ with an arbitrary "test function" $f$ (i.e., $f$ is a smooth function of compact support) so as to define

$$\phi(f) = \int f(t, \vec{x})\ \phi(t, \vec{x})\ d^4 x \qquad \text{rather than} \qquad \phi(t, \vec{x})$$

the resulting formula for $\phi(f)$ can be shown to make rigorous mathematical sense

**thus defining $\phi$ as an 'operator-valued distribution'**

# reminder

- ground state $|0\rangle$ of $\Phi$ is simply simultaneous ground state of all of the harmonic oscillators that comprise $\Phi$ , i.e., it is the state satisfying $a_{\vec{k}}\,|0\rangle = 0$ for all $\vec{k}$

- in quantum field theory this state is interpreted as representing the "vacuum"

- state of the form $(a^{+})^{n}\,|0\rangle$ interpreted as a state where a total of n particles are present

- in an interacting theory, the state of the field may be such that the field behaves like a free field at early and late times

→ in that case, we would have a particle interpretation of the states of the field at early and late times

→ relationship between the early and late time particle descriptions of a state – given by the S-matrix – contains a great deal of dynamical information about the interacting theory

(and, indeed, contains all of the information relevant to laboratory scattering experiments)

# reminder

- $\Phi$ and $\pi$ operators in the Heisenberg picture

$$\phi(\boldsymbol{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a_{\boldsymbol{p}}\, e^{-ip\cdot x} + a^+_{\boldsymbol{p}}\, e^{ip\cdot x}\right) \Big|_{p^0=E_{\boldsymbol{p}}}$$

$$\pi(\boldsymbol{x},t) = \frac{\partial}{\partial t}\, \phi(\boldsymbol{x},t)$$

- this equation makes explicit the dual particle and wave interpretation of the quantum field $\Phi$ (x)

→ on the one hand $\Phi$(x) is written as a Hilbert space operator which creates and destroys the particles that are the quanta of field excitation

→ on the other hand $\Phi$(x) is written as a linear combination of solutions ($e^{ip.x}$ and $e^{-ip.x}$) of the Klein-Gordon equation.

both signs of the time dependence in the exponential appear
i.e. both $e^{-ip^0 t}$ and $e^{+ip^0 t}$ although $p^o$ is always positive

→ if these were single-particle wavefunctions
they would correspond to states of positive and negative energy
let us refer to them more generally as *positive- and negative-frequency modes*

# reminder

- the connection between the particle creation operators and the waveforms displayed here is always valid for free quantum fields:

→ **a positive-frequency solution of the field equation has as its coefficient the operator that destroys a particle in that single-particle wavefunction**

→ **a negative-frequency solution of the field equation, being the Hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that creates a particle in that positive-energy single-particle wavefunction**

- in this way the fact that relativistic wave equations have both positive- and negative-frequency solutions is reconciled with the requirement that a sensible quantum theory contain only positive excitation energies.

# Penrose diagram

- a **conformal transformation** which brings entire manifold onto a **compact** region such that we can fit the spacetime (ie. its infinities) on a finite 2-dimensional diagram

→ known as Penrose-Carter diagram
(or Carter-Penrose diagram or just conformal diagram)

→ make infinity were a « definite place »

# Penrose diagram for Minkowski space

- idea → introduce a transformation that takes Minkowski space into a compact region

- begin with the line element in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2\left( d\theta^2 + \sin^2\theta \; d\phi^2 \right)$$

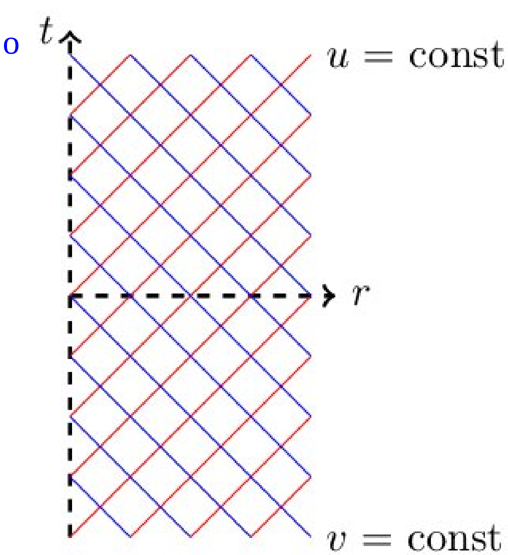
- now define : u = t − r and v = t + r then we have :

$$ds^2 = -du\,dv + \frac{1}{4}(u - v)^2 \left(d\theta^2 + \sin^2 d\phi^2\right)$$

→ this is a rotation of the *t and r*-axes to the *u and v*-axes by 45 $^{o}$

→ radial light rays (ds = 0 and dθ = dΦ = 0) give dudv = 0

→ thus radial light rays travel on lines of constant u and v



# Penrose diagram for Minkowski space

- now let

$$u' = \tan^{-1} u = \frac{1}{2}(\tau - \rho)$$

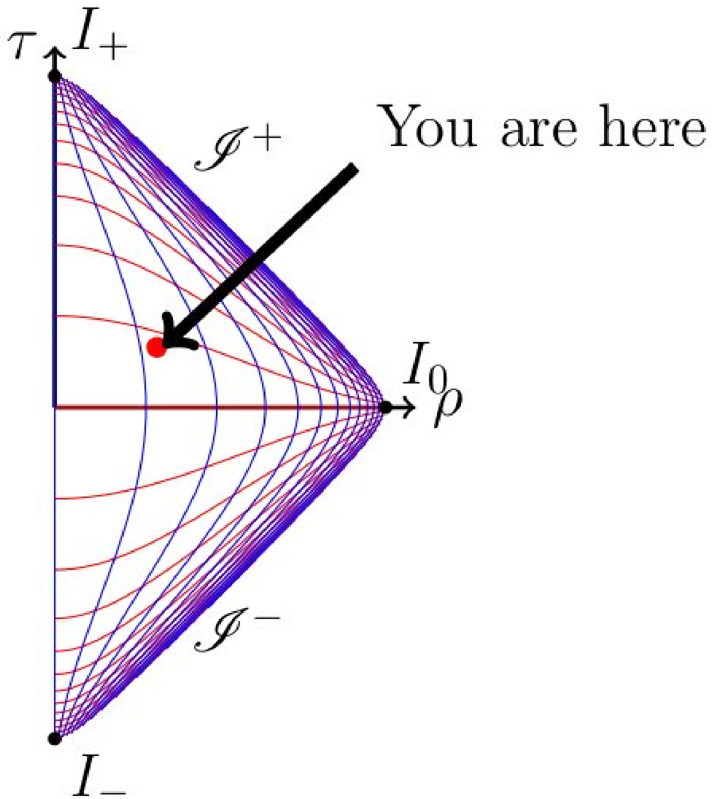
$$v' = \tan^{-1} v = \frac{1}{2}(\tau + \rho)$$

since $0 < r < \infty$ and $-\infty < t < \infty$ then $-\pi/2 < u$ and $v < \pi/2$

- rotating this primed system by 45° we obtain a set of new coordinates

$$\tau = \tan^{-1} u + \tan^{-1} v = \tan^{-1}(t - r) + \tan^{-1}(t + r)$$

$$\rho = \tan^{-1} v - \tan^{-1} u = \tan^{-1}(t + r) - \tan^{-1}(t - r)$$




→ these give the Penrose diagram for Minkowski space

→ lines of constant r and t are shown in the diagram

# Penrose diagram for Minkowski space

- **we have mapped infinity to a finite region**

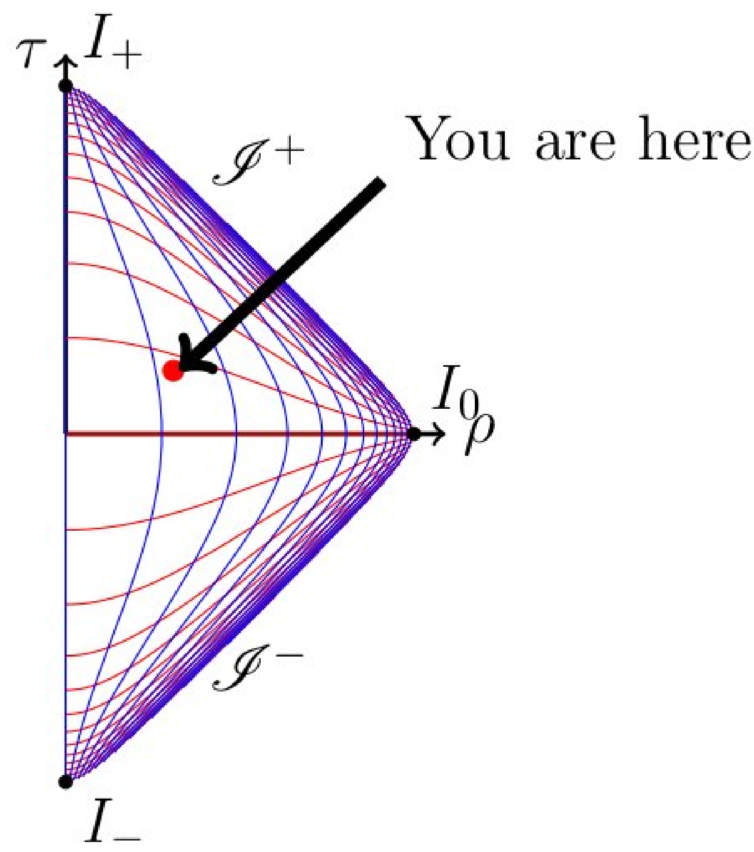
→ there are several types of infinity:
$I_+$ , $I_-$ , $I_0$ and $\mathfrak{I}^{\pm}$

- outgoing light rays follow paths $t = r + \text{const.}$

→ they leave along lines of slope 1

→ they arrive at $v' = \pi/2$ or future **null infinity** $\mathfrak{I}_+$

this symbol $\mathfrak{I}^+$ is called scri plus




- ingoing radial lines end at $\mathfrak{I}^-$ i.e. past null infinity

- the motion of particles start at **past timelike infinity** $I_-$
and end at **future timelike infinity** $I_+$

- finally, spacelike trajectories arrive at **spacelike infinity** $I_0$

# Schwarzschild solution

solution of Einstein equations describing the exterior gravitational field of a static and spherically symmetric body (M = Gm)

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with : $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

- metric singularity at r = 0 is a **true singularity**

  i.e. a singularity of the spacetime geometry (curvature scalars blow up)

- metric singularity at r = 2M is an **apparent singularity**

  i.e. not a singularity of the spacetime geometry (no curvature scalars blow up)

  coordinates fail to properly cover a region of spacetime

  depends on the coordinate frame we use and has no physical significance

  sometimes referred to as a coordinate singularity or a ‘breakdown’ of coordinates

# Schwarzschild solution

- solution of Einstein equations describing the exterior gravitational field of a static and spherically symmetric body (M = Gm) :

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = -\left(1-\frac{2M}{r}\right) dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with : $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

- introducing (Regge Wheeler tortoise coordinate) :

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

- concentrating on the r and t parts the metric becomes :

$$ds^2 = \left(1-\frac{2M}{r}\right)\left[-dt^2 + dr^{*2}\right]$$

# Schwarzschild solution

- moving to null coordinates by writing : $u = t - r^*$ and $v = t + r^*$

→ the metric becomes :

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv$$

→ or

$$ds^2 = -\frac{2M\, e^{-r/2M}}{r}\, e^{(v-u)/4M}\, dudv$$

→ or, using $U = -e^{-\frac{u}{4M}}$ and $V = e^{\frac{v}{4M}}$ (U < 0 and V > 0 for all values of r) :

$$ds^2 = -\frac{32 M^3 e^{-r/2M}}{r}\, dUdV$$

→ metric well defined for $r = 2M$ (no singularity) i .e. $U = 0$ or $V = 0$, and for all $r > 0$

→ can thus extend the Schwarzschild solution by allowing $U$ and $V$ to take on all values compatible with $r > 0$

# Schwarzschild solution

- make the final transformation $T = (U + V)/2$ and $X = (V - U)/2$ (or $U = T - X$ , $V = T + X$) the full Schwarzschild metric takes the final form given by Kruskal and Szekeres :

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}\left(-dT^2 + dX^2\right) + r^2 d\Omega^2$$

- relation between the old coordinates (t, r) and the new coordinates (T, X) given by

$$\left(\frac{r}{2M} - 1\right) e^{-r/2M} = X^2 - T^2$$

$$\frac{t}{2M} = \ln\left(\frac{T + X}{X - T}\right) = 2\tanh^{-1}\left(T/X\right)$$

- this spacetime is called the « extended black hole spacetime » or also « extended Schwarzschild geometry »

# Schwarzschild solution

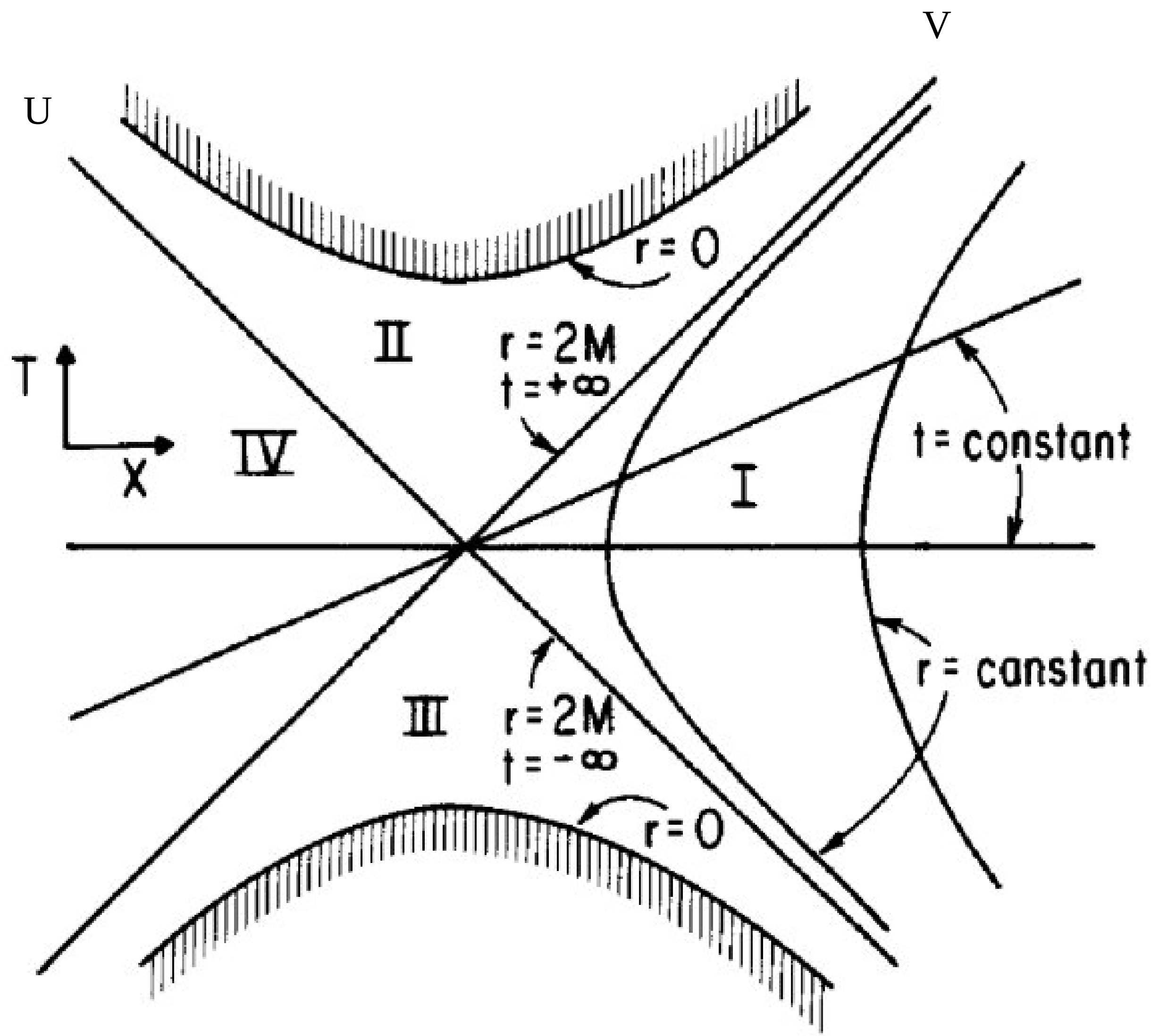
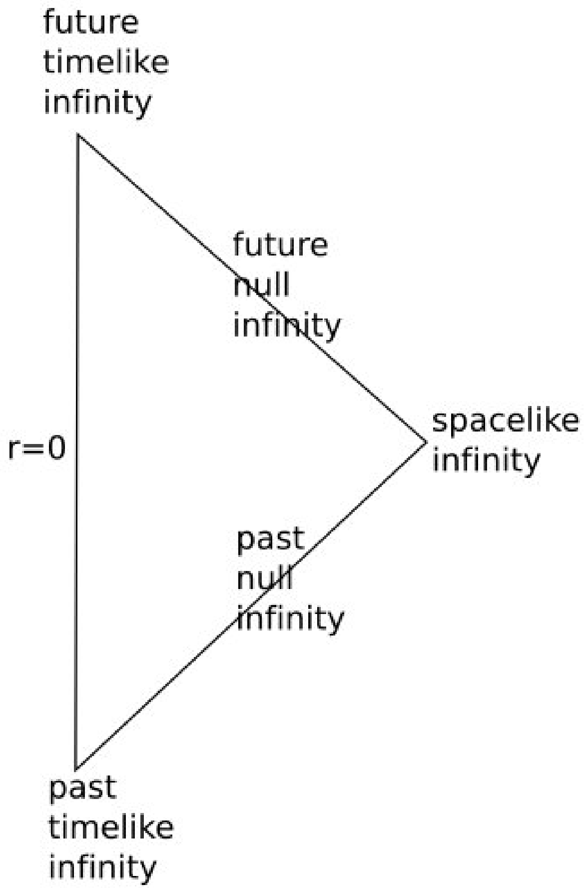
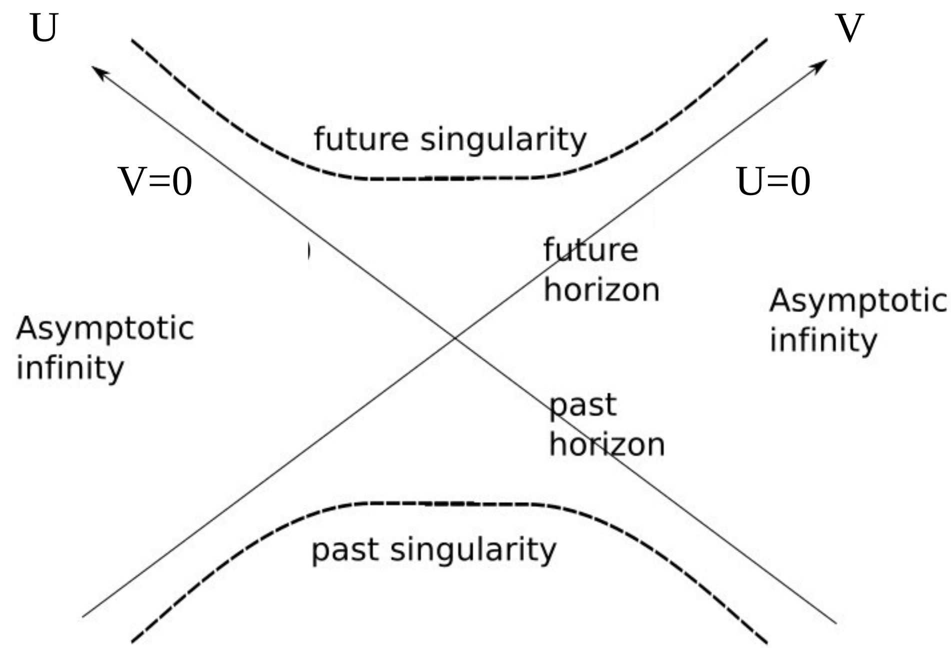



Fig. 6.9. The Kruskal extension of Schwarzschild spacetime.

R. Wald : General Relativity, The University of Chicago Press

# Penrose Diagram




Penrose diagram of Minkowski space

fully extended Schwarzschild geometry
(all values of U and V)

# Schwarzschild solution

- at the horizon $r = 2M$ we have $UV = 0$ $\rightarrow$ either $U = 0$ or $V = 0$

- singularity $r = 0$ corresponds to the (two branches of the) hyperbola described by $UV = 1$

  $\rightarrow$ represented by a wavy line (singularities will always be represented by wavy lines)

- in general, surfaces of $r = const.$ correspond to hyperbolae $UV = const.$ with $UV < 1$

- spatial sections with $t = const.$ have $U/V = const.$ and $|U/V| < 1$

- ingoing and outgoing null geodesics are respectively given by $U = const.$ and $V = const.$

# Schwarzschild solution

- the U, V coordinates cover all of our spacetime but these coordinates do not have a bounded range

- thus if we try to draw the U, V space on a sheet of paper, we have to stop at a finite value of U, V , and we do not explicitly see the picture of how the 'points at infinity' border our spacetime

- to bring these 'points at infinity' to a finite coordinate distance from the points in the interior of our spacetime, we make a conformal rescaling of the metric

- here the word 'conformal' means that at each point the metric is scaled by a number $g_{ab}(x) \rightarrow \Omega^2(x)\, g_{ab}(x)$ so that the angles between different directions at the point x do not change and in particular null directions remain null directions

- such a rescaling helps to understand the causal structure of the spacetime including the behavior of 'infinity'

# Schwarzschild solution

- define a new set of null coordinates via $U = \tan \tilde{U}$ and $V = \tan \tilde{V}$
such that $-\pi/2 < \tilde{U}\,,\ \tilde{V} < \pi/2$

→ the line-element

$$ds^2 = -\frac{32M^3 e^{-r/2M}}{r}\, dU dV \qquad \text{or} \qquad ds^2 = -\frac{32M^3 e^{-r/2M}}{r}\, dU dV + r^2 d\Omega^2$$

putting back the angular variables part

becomes

$$ds^2 = \left(2\cos\widetilde{U}\cos\widetilde{V}\right)^{-2}\left[-4\frac{32M^3 e^{-r/2M}}{r}\, d\widetilde{U}\, d\widetilde{V} + r^2\cos^2\widetilde{U}\cos^2\widetilde{V}\, d\Omega^2\right]$$

- performing the conformal transformation we get :

$$d\widetilde{s}^2 = \left(2\cos\widetilde{U}\cos\widetilde{V}\right)^{2} ds^2$$

$$= -4\frac{32M^3 e^{-r/2M}}{r}\, d\widetilde{U}\, d\widetilde{V} + r^2\cos^2\widetilde{U}\cos^2\widetilde{V}\, d\Omega^2$$

and we add the points at infinity

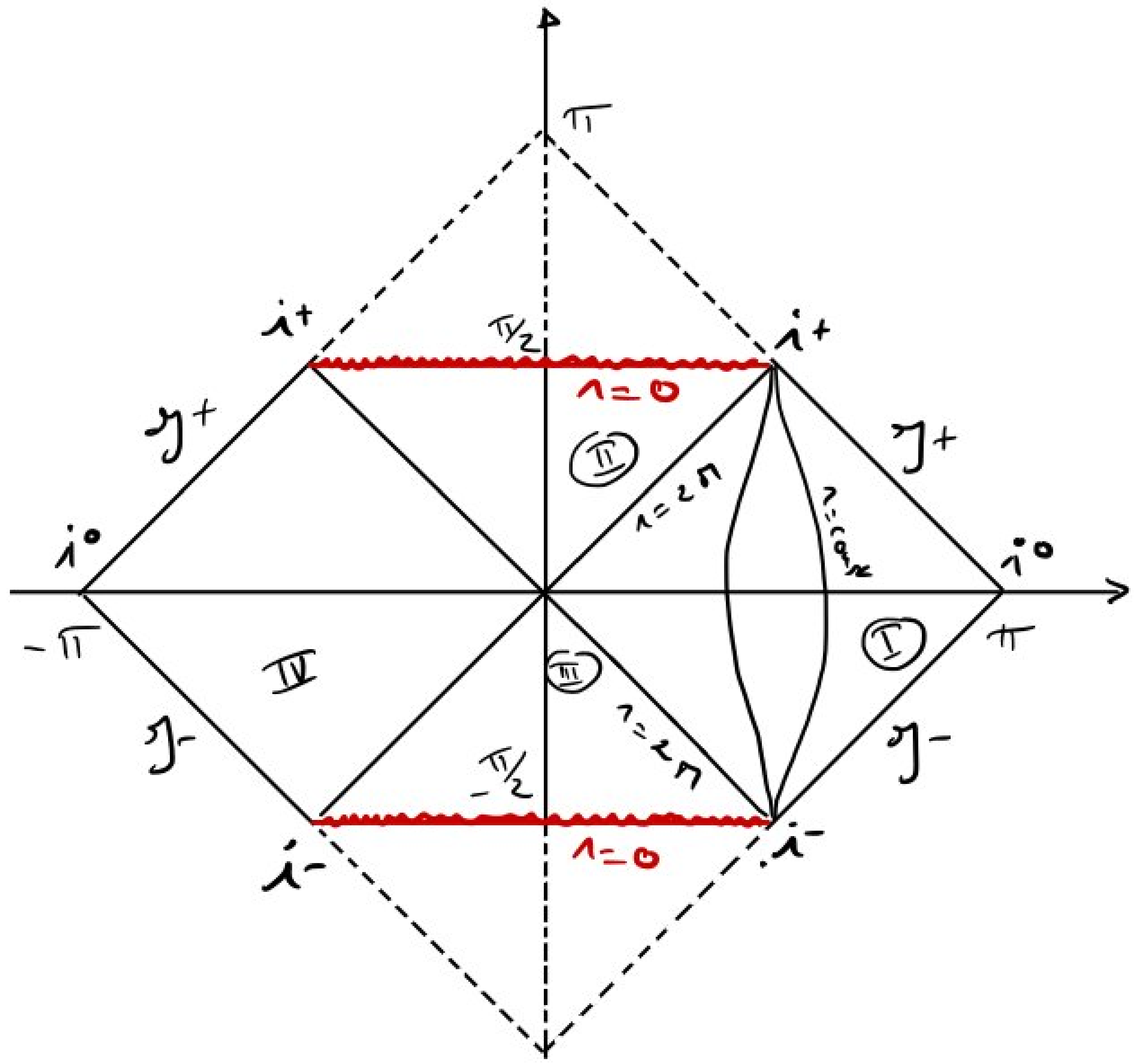
# Schwarzschild solution

- the curvature singularity *UV = 1* now corresponds to

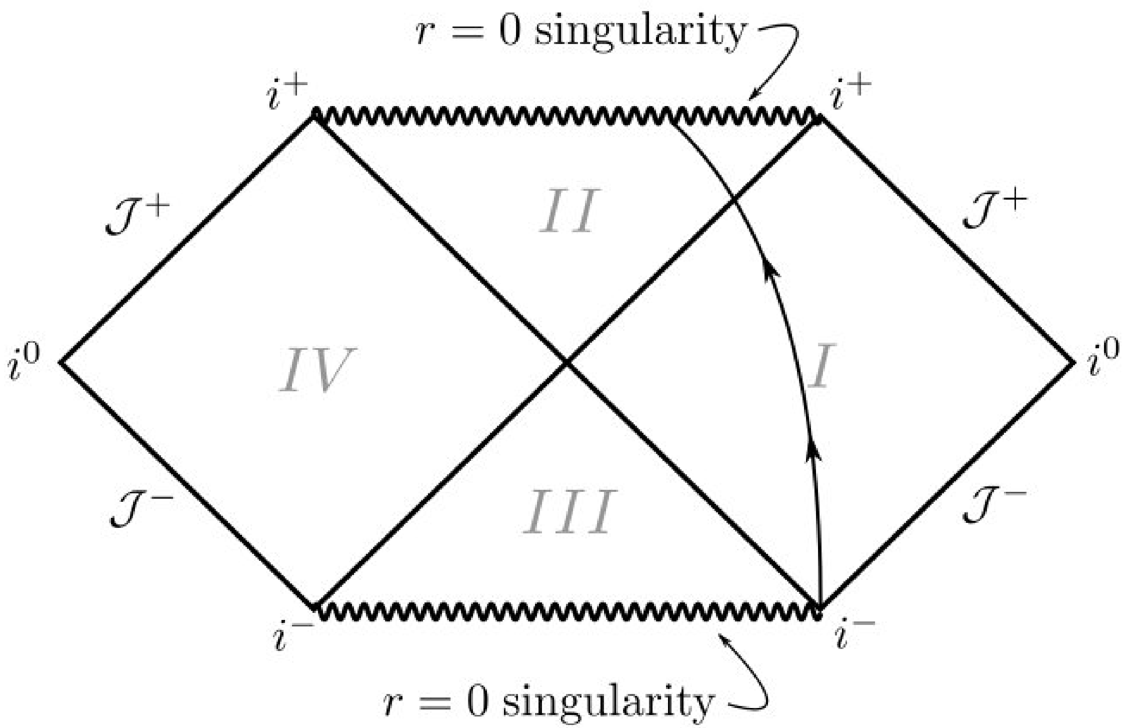
$$\tan\tilde{U}\tan\tilde{V} = 1 \Leftrightarrow \sin\tilde{U}\sin\tilde{V} = \cos\tilde{U}\cos\tilde{V} \Leftrightarrow \cos\left(\tilde{U} + \tilde{V}\right] = 0$$

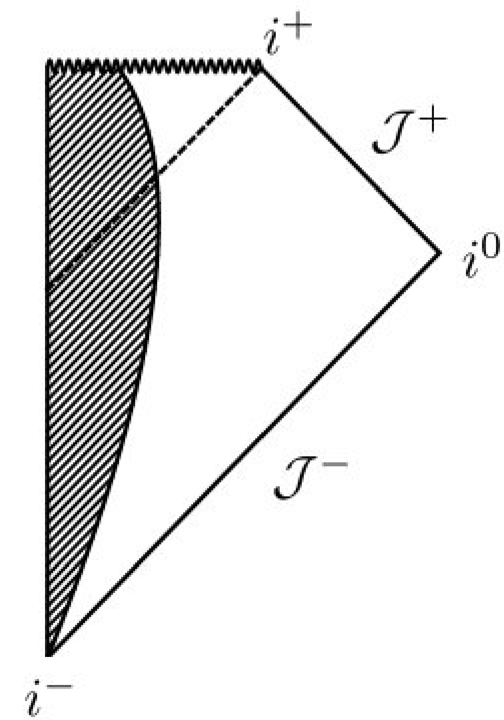
which implies $\tilde{U} + \tilde{V} = \pm\pi/2$ or $\tilde{T} = \pm\pi/4$
if we define $\tilde{T}$ and $\tilde{X}$ through $\tilde{U} = \tilde{T} - \tilde{X}$ and $\tilde{V} = \tilde{T} + \tilde{X}$

# Penrose Diagram



# Penrose Diagram



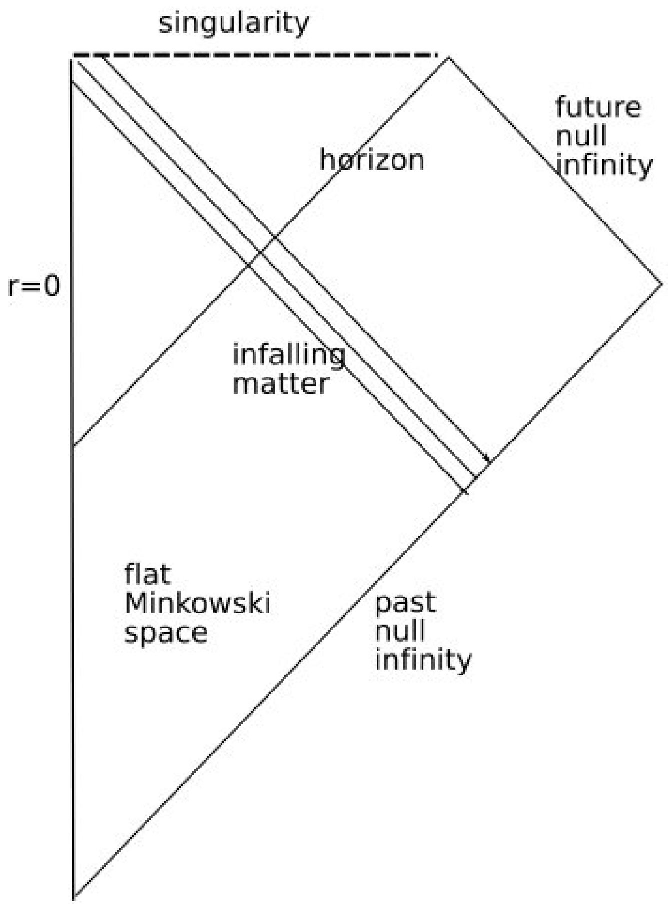

Penrose diagram for Kruskal space

Penrose diagram for a collapsing star

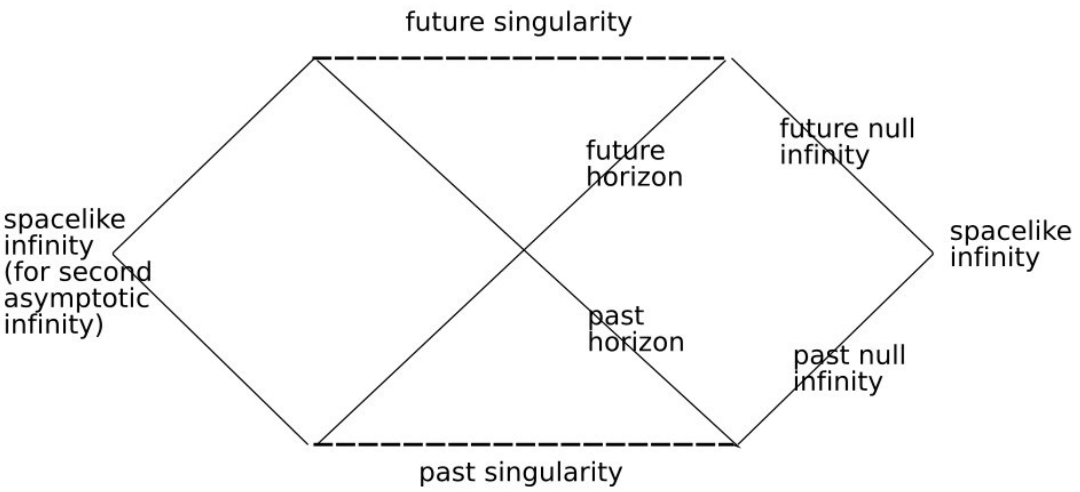
curved line represents the surface and the shaded region corresponds to the interior of the star

horizon corresponds to the dashed line

# Penrose Diagram




Penrose diagram of the black hole
made by collapse of a shell




Penrose diagram for the ‘eternal Schwarzschild hole’

# Penrose Diagram

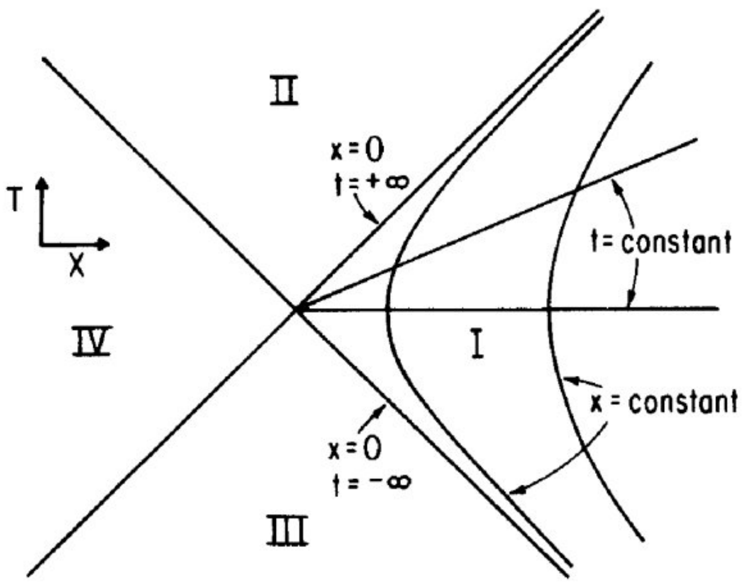



Fig. 6.8. Rindler spacetime, displayed as the "wedge," I, of two-dimensional Minkowski spacetime.

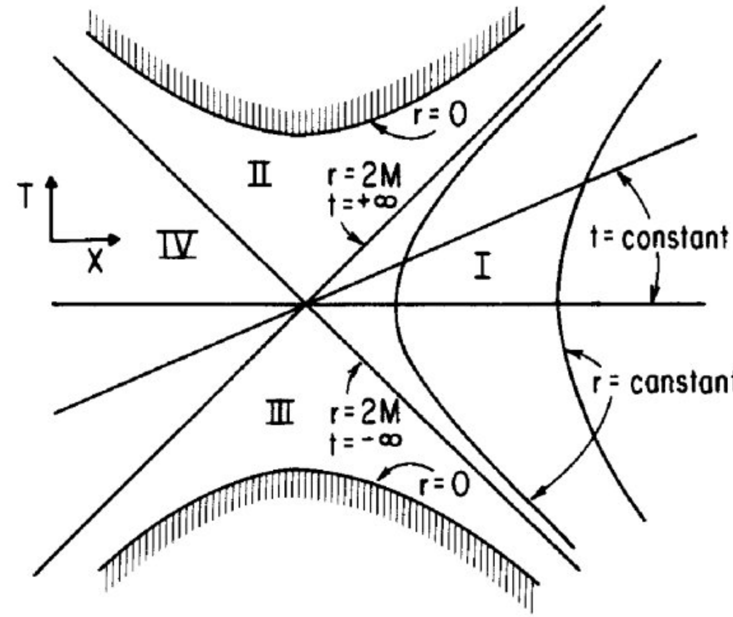



Fig. 6.9. The Kruskal extension of Schwarzschild spacetime.

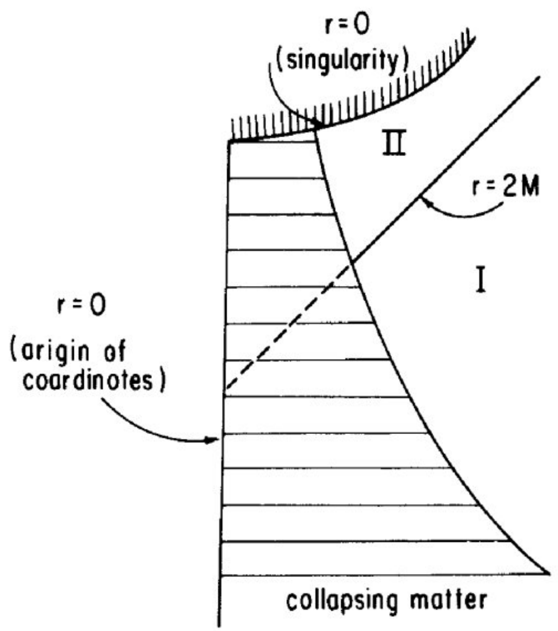



Fig. 6.11. The spacetime resulting from the complete gravitational collapse of a spherical body. All of regions III and IV of the extended Schwarzschild spacetime (Fig. 6.9) are "covered up" by the collapsing matter. However, (part of) the black hole region II is produced.

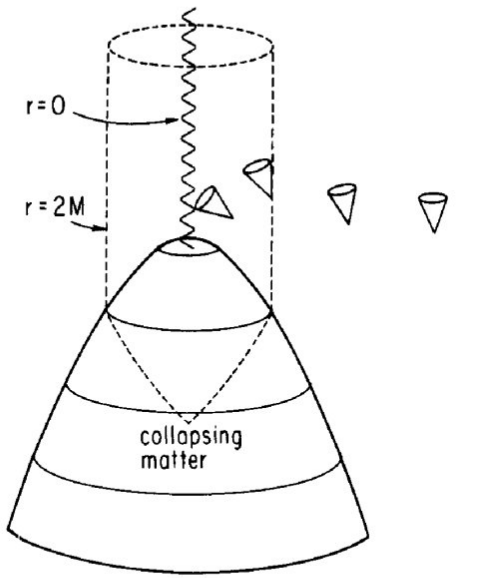



Fig. 6.12. Another representation of the spacetime of Figure 6.11. Here, one of the two suppressed spatial dimensions is restored, so each of the circles shown on the collapsing body corresponds to the 2-sphere surface of the body at an instant of time. However, the light cones no longer are represented by 45° lines. Indeed, the spacelike nature of the singularity and the inevitable capture by the singularity of any particle or light ray in the region $r < 2M$ is illustrated here by the "tipping over" of the future light cones in the strong field region.

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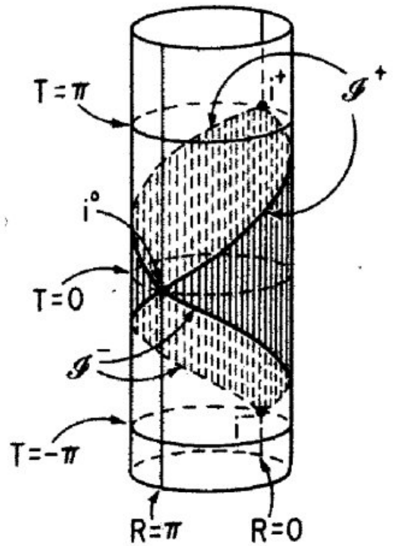


Fig. 11.1. A spacetime diagram of the Einstein static universe. As described in the text, Minkowski spacetime is conformally isometric to the region $O = I^+(i^-) \cap I^-(i^+)$ of this spacetime. The boundary of $O$—consisting of the points $i^-$, $i^+$, and $i^0$ and the null hypersurfaces $\mathscr{I}^-$ and $\mathscr{I}^+$—defines a precise notion of "infinity" for Minkowski spacetime.

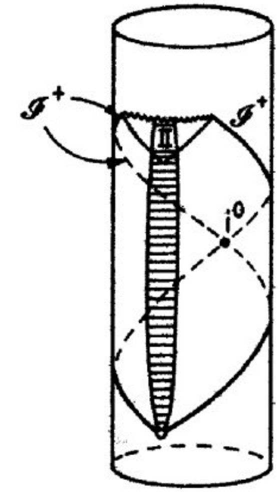


Fig. 12.1. A conformal diagram of the same spacetime as shown in Figures 6.11 and 6.12. From this conformal diagram, it is apparent that region II of the physical spacetime lies outside of $J^-(\mathscr{I}^+)$. In contrast, in Figure 11.1, $J^-(\mathscr{I}^+)$ includes the entire physical spacetime.

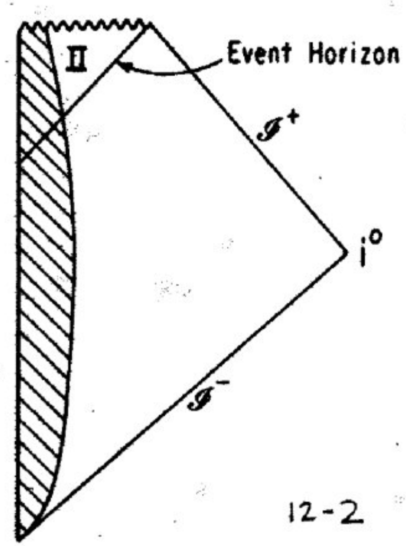


Fig. 12.2. Another representation of the closure, $\overline{M}$, of the physical spacetime depicted in Figure 12.1. As in Figure 12.1, the angular dimensions are suppressed so each point in this diagram (except those at $r = 0$ and the point $i^0$) represents a 2-sphere.

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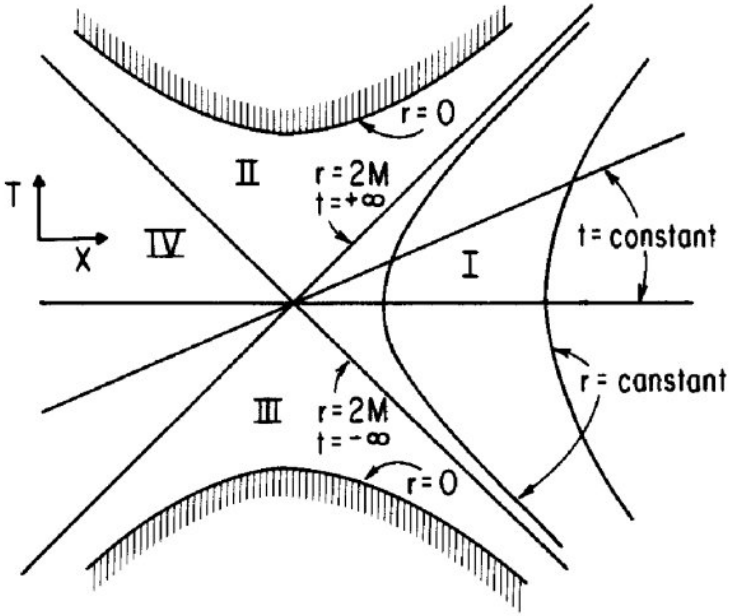
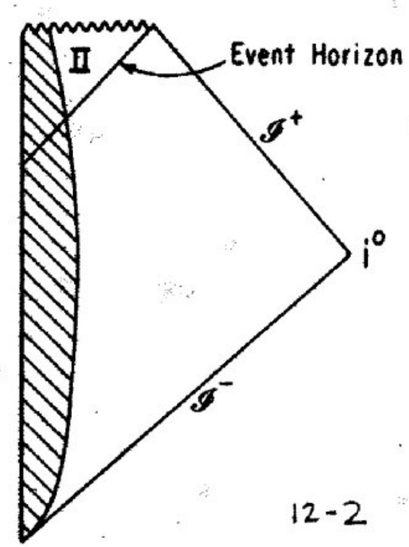



Fig. 6.9. The Kruskal extension of Schwarzschild spacetime.

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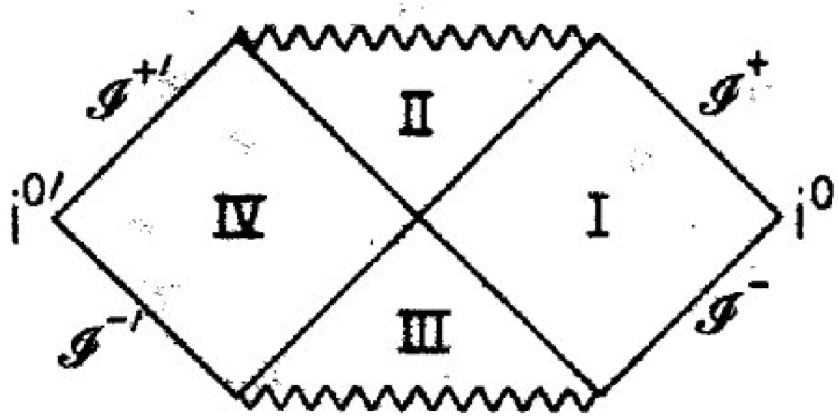


Fig. 12.3. A conformal diagram of the extended Schwarzschild spacetime (see Fig. 6.9), represented in the same manner as used in Figure 12.2. Note that since the extended Schwarzschild spacetime has two distinct asymptotically flat regions, two distinct conformal boundaries are shown.

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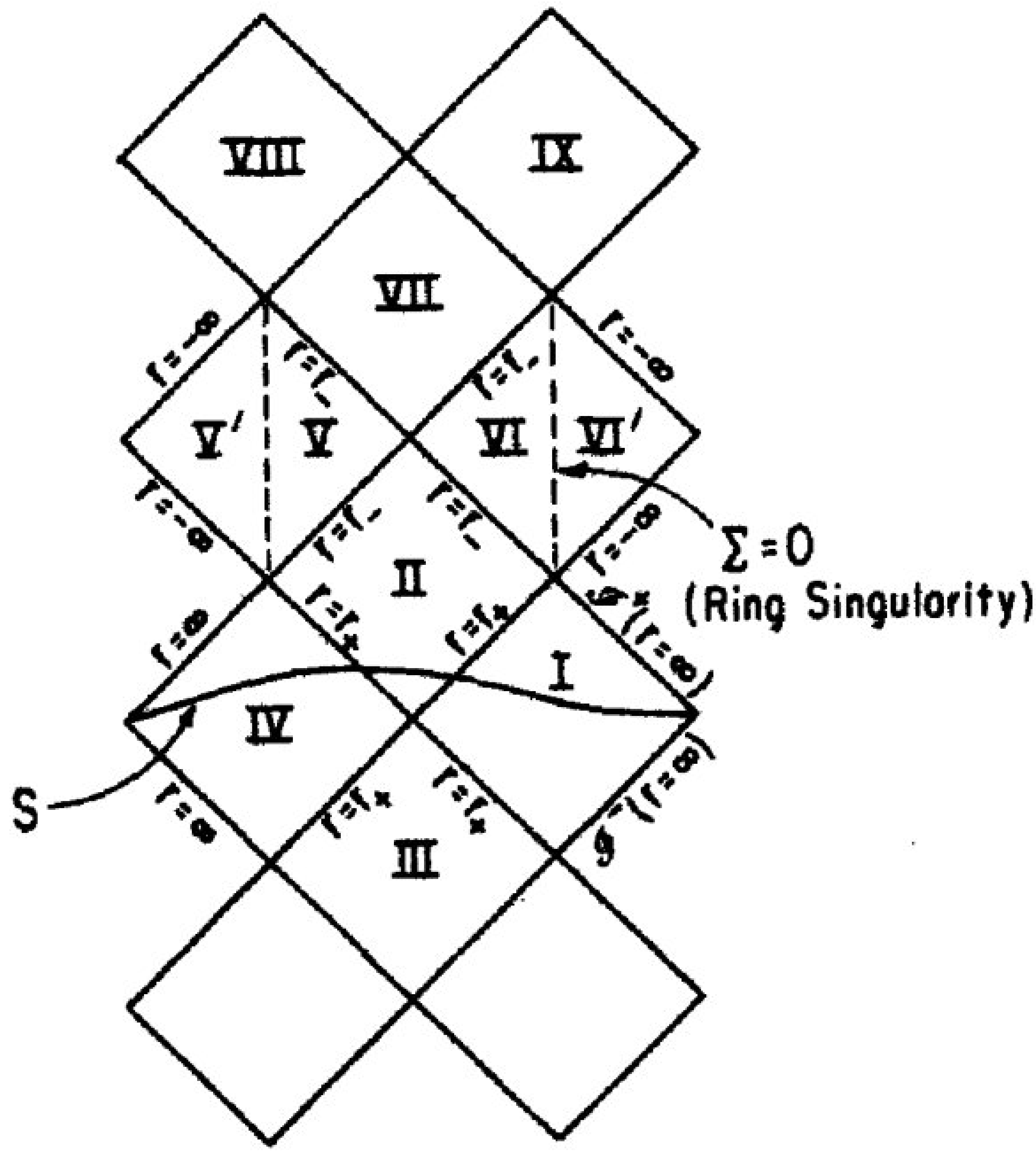


**Fig. 12.4.** A conformal diagram of the extended charged Kerr spacetime in the case $a \neq 0$, $a^2 + e^2 < M^2$.

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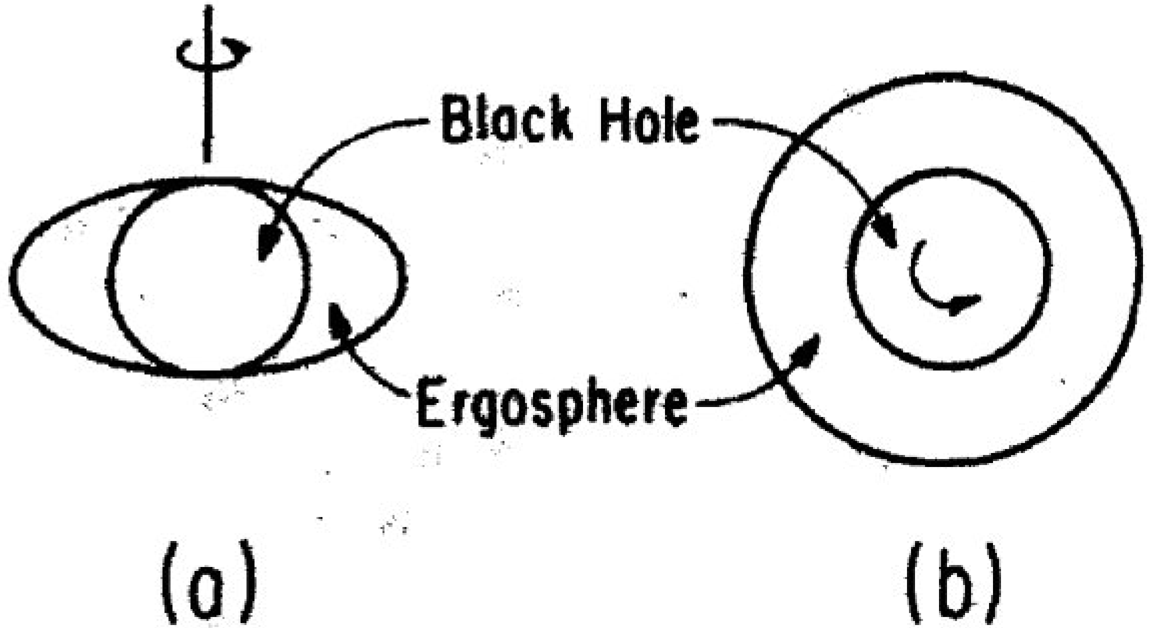


**Fig. 12.6.** A sketch showing (*a*) a "side view" and (*b*) a "top view" of the ergosphere of a Kerr black hole.

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